We prove a character formula for any finite-dimensional irreducible representation $V$ of the “queer” Lie superalgebra $g = q(n)$. It expresses $\text{ch} V$ in terms of the multiplicities of the irreducible $g$-subquotients of the cohomology groups of certain dominant $g$-bundles on the $\Pi$-symmetric projective spaces (i.e., on the homogeneous superspaces $G/P$ whose reduced space is a projective space, where $G = Q(n)$). We also establish recurrent relations for the above multiplicities, and this enables us to compute explicitly $\text{ch} V$ for any given $V$. This provides a complete solution to the Kac character problem for the Lie superalgebra $q(n)$. Finally, we consider the particular cases of $q(2)$, $q(3)$, and $q(4)$ in which we compare the new character formula with the generic character formula of [12].

Introduction

In this paper, we solve the Kac character problem posed in [5], i.e., the problem of computing the character of any irreducible finite-dimensional representation, for the Lie superalgebra $g = q(n)$. Our solution is based on the same general ideas as the second author’s solution of the Kac character problem for $gl(m|n)$, [16], but in the case of $q(n)$ the argument is almost entirely geometric. The homogeneous superspaces we consider are Manin’s $\Pi$-symmetric flag supermanifolds. In 1982, Manin constructed the flag supermanifolds corresponding to all classical series of simple Lie supergroups, and in particular constructed the $\Pi$-symmetric flag superspaces corresponding to $Q(n)$ (or to the simple Lie supergroup $PSQ(n)$), see [6, 7]. The standard reference today is Manin’s monograph [8]. Immediately after constructing the flag supermanifolds (of all types corresponding to the different series of classical simple Lie superalgebras), Manin formulated the problem of finding an analog of Borel–Weil–Bott’s theorem (or theory) for this case. It was quite clear that the cohomology of the flag supermanifolds deserves by itself to be calculated, but Manin’s hope was that this cohomology should also give an approach to calculating the characters of the irreducible representations.

This problem of Manin turned out to be a difficult one. Some progress was made during the 80’s (see, for example, [11]) where, roughly speaking, a Borel–Weil–Bott-type theorem was proved for typical representations. Later (see [13] and [12]) the theory (in a general $D$-module version inspired by the celebrated work of Beilinson and Bernstein) was extended to generic representations. Finite-dimensional singly atypical (but not necessarily generic) $gl(m|n)$-modules have been studied in [3]. However, the case of arbitrary finite-dimensional irreducible representations remained essentially intractable until [15, 16], where Kac’s character problem was solved for $gl(m|n)$ by a mixture of algebraic and geometric techniques. The method developed there by the second author enables us to carry out Manin’s program also for the $\Pi$-symmetric projective superspaces and in particular to give a geometry-based complete solution of Kac’s character problem for $g = q(n)$. 

Let us describe the contents of the paper. The objective of Sec. 1 is to present and explain the results. In Secs. 1.1 and 1.2, we fix the notations. In Sec. 1.3, we state our main results in four theorems and two corollaries. The character formula of Theorem 1 reduces the problem of calculating the character of an irreducible finite-dimensional $g$-module to calculating the multiplicities of the irreducible $g'$-subquotients (where $g' = q(n - k)$) of the cohomologies of dominant $g'$-linearized bundles on the $\Pi$-symmetric projective superspaces of $G' = Q(n - k)$ for $k = 1, \ldots, n - 2$. Theorems 2 and 3 establish recurrent relations which reduce the calculation of the above multiplicities to the calculation of the multiplicities of the trivial irreducible
representation in the cohomologies of a certain bundle (corresponding to the highest weight of the adjoint representation) on the \( \Pi \)-symmetric projective superspace. Theorem 4 calculates the latter multiplicities explicitly. Together, Theorems 1–4 provide a complete solution to the Kac character problem for \( g = q(n) \). Section 1.4 is devoted to examples: we compute explicitly the characters of all irreducible finite-dimensional \( q(2) \), \( q(3) \), and \( q(4) \)-modules and compare the results with the approximation given by the generic character formula of [12].

Section 1 contains practically no proofs. Since the proofs are quite technical, we have chosen to present them in a separate section. This is Sec. 3. In Sec. 2, we prove some auxiliary results which are needed in the proofs of the main results.

Acknowledgment. Part of this work was done at the Erwin Schrödinger Institute in Vienna during both authors' visit there in April and May 1996. The paper was completed in the summer of 1996, shortly after the first author's stay at the Max-Planck-Institut für Mathematik in Bonn. We thank both institutions for their support and hospitality. The first author acknowledges also partial NSF support throughout his work on this project. The second author acknowledges the support of the Sloan Foundation.

1. Preliminaries and Statement of Main Results

1.1. Algebraic preliminaries. The ground field is \( \mathbb{C} \). All vector spaces are automatically assumed to be \( \mathbb{Z}_2 \)-graded and a subscript 0 or 1 (to any \( \mathbb{Z}_2 \)-graded object such as vector space or sheaf) always refers to the \( \mathbb{Z}_2 \)-grading. \( \Pi \) denotes the functor of parity change. The dimension of a vector space \( V = V_0 \oplus V_1 \) is by definition \( k + \ell \epsilon \), where \( k = \dim V_0 \), \( \ell = \dim II V_1 \), and \( \epsilon \) is a formal odd variable with \( \epsilon^2 = 1 \). One has dim \( II V = \epsilon \cdot \dim V \). If \( \dim V = k + \ell \epsilon \), we set \( \dim V := k + l \). The superscript * denotes dual space.

Throughout this paper, \( E \) will denote a fixed vector space of dimension \( n + n \epsilon \), \( n \geq 2 \). \( Q(E) \) is the Lie supergroup of endomorphisms of \( E \) which preserve a given odd isomorphism \( \Pi_E : E \cong E \) with \( \Pi^2_E = \text{id} \). \( q(E) \) is the Lie superalgebra of \( Q(E) \), i.e., \( q(E) = \text{Lie} G \), where \( \text{Lie} \) denotes the Lie superalgebra functor. \( E \) is the tautological representation of \( g \) and \( G \).

Let \( \mathfrak{h} \) be a fixed Cartan subsuperalgebra of \( g \), i.e., a nilpotent self-normalizing Lie subsuperalgebra of \( g \) (see, for example, [14]). Then \( \dim \mathfrak{h} = n + n \epsilon \) and \( \mathfrak{h}_0 \) is a Cartan subalgebra of \( \mathfrak{g}_0 = gl(\mathfrak{e}_0) \). The roots \( \Delta \) of \( g \) (see [14]) are nothing but the roots of \( gl(\mathfrak{e}_0) \):

\[ \Delta = \{ \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n \}; \]

\( \epsilon_1, \ldots, \epsilon_n \) being a standard basis in \( \mathfrak{h}_0^* \). For each \( \alpha \in \Delta \) the dimension of the root space \( g(\alpha) = 1 + \epsilon \). \( W \) denotes the Weyl group of \( \mathfrak{g}_0 \). (\( W \) is a symmetric group of order \( n \).) The weights are by definition the elements of \( \mathfrak{h}_0^* \). If \( \lambda \in \mathfrak{h}_0^* \), we will usually write \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where the standard coordinates \( \lambda_i \) of \( \lambda \) are its coordinates with respect to \( \epsilon_i \), i.e., \( \lambda = \sum_{i=1}^{n} \lambda_i \epsilon_i \). We also set \( \# \lambda := \# \{ i \mid \lambda_i \neq 0 \} \). \( \Lambda = \{ \lambda \in \mathfrak{h}_0^* \mid \lambda_i - \lambda_j \in \mathbb{Z} \forall i, j, 1 \leq i, j \leq n \} \) is the set of integral weights. We say that a weight \( \lambda \) is a reduced expression of a weight \( \tilde{\lambda} \) if \( \tilde{\lambda} \) is obtained from \( \lambda \) by replacing a maximal number of pairs of coordinates \( \lambda_i, \lambda_j \) with \( \lambda_i + \lambda_j = 0 \) by pairs of the form \( 0, 0 \). For instance, when \( n = 5 \), \( (1, 1, 1, -1, -1) \) is a weight and all its reduced expressions are \( (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), \) and \( (0, 0, 0, 0, 0) \). If \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \alpha = \epsilon_i - \epsilon_j \), we say that \( \lambda \) is \( \alpha \)-typical when \( \lambda_i + \lambda_j \neq 0 \) and that \( \lambda \) is \( \alpha \)-atypical when \( \lambda_i + \lambda_j = 0 \). The set of all \( \alpha \)-atypical weights will be denoted by \( \mathfrak{h}_0^{\alpha} \); \( \Lambda_{\alpha} = \Lambda \cap \mathfrak{h}_0^{\alpha} \).

\( Z \) denotes the center of the enveloping algebra \( U(\mathfrak{g}) \). \( Z \) is a commutative \( \mathbb{C} \)-algebra and there is a canonical injective algebra homomorphism, the Harish-Chandra homomorphism

\[ HC : Z \hookrightarrow S(\mathfrak{h}_0)^W \]