SOME PROBLEMS OF 'PARTITIO NUMERORUM'; III: ON THE EXPRESSION OF A NUMBER AS A SUM OF PRIMES.

BY

G. H. HARDY and J. E. LITTLEWOOD.

New College, OXFORD. Trinity College, CAMBRIDGE.

1. Introduction.

1. 1. It was asserted by GOLDBACH, in a letter to EULER dated 7 June, 1742, that *every even number 2m is the sum of two odd primes*, and this proposition has generally been described as 'Goldbach's Theorem'. There is no reasonable doubt that the theorem is correct, and that the number of representations is large when m is large; but all attempts to obtain a proof have been completely unsuccessful. Indeed it has never been shown that every number (or every *large* number, any number, that is to say, from a certain point onwards) is the sum of 10 primes, or of 1 000 000; and the problem was quite recently classified as among those 'beim gegenwärtigen Stande der Wissenschaft unangreifbar'.[1]

In this memoir we attack the problem with the aid of our new transcendental method in 'additiver Zahlentheorie'.[2] We do not solve it: we do not

[1] E. LANDAU, 'Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zetafunktion', *Proceedings of the fifth International Congress of Mathematicians*, Cambridge, 1912, vol. 1, pp. 93—108 (p. 105). This address was reprinted in the *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 21 (1912), pp. 208—228.

[2] We give here a complete list of memoirs concerned with the various applications of this method.

G. H. HARDY.

1. 'Asymptotic formulae in combinatory analysis', *Comptes rendus du quatrième Congrès des mathematiciens Scandinaves à Stockholm*, 1916, pp. 45—53.

2. 'On the expression of a number as the sum of any number of squares, and in particular of five or seven', *Proceedings of the National Academy of Sciences*, vol. 4 (1918), pp. 189—193.

even prove that any number is the sum of 1 000 000 primes. In order to prove anything, we have to assume the truth of an unproved hypothesis, and, even on this hypothesis, we are unable to prove Goldbach's Theorem itself. We show, however, that the problem is not 'unangreifbar', and bring it into contact with the recognized methods of the Analytic Theory of Numbers.

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3. 'Some famous problems of the Theory of Numbers, and in particular Waring's Problem' (Oxford, Clarendon Press, 1920, pp. 1—34).

4. 'On the representation of a number as the sum of any number of squares, and in particular of five', *Transactions of the American Mathematical Society*, vol. 21 (1920), pp. 255—284.

5. 'Note on Ramanujan's trigonometrical sum $c_q(n)$', *Proceedings of the Cambridge Philosophical Society*, vol. 20 (1921), pp. 263—271.

G. H. Hardy and J. E. Littlewood.

1. 'A new solution of Waring's Problem', *Quarterly Journal of pure and applied mathematics*, vol. 48 (1919), pp. 272—293.

2. 'Note on Messrs. Shah and Wilson's paper entitled: On an empirical formula connected with Goldbach's Theorem', *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 245—254.

3. 'Some problems of 'Partitio numerorum'; I: A new solution of Waring's Problem', *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen* (1920), pp. 33—54.

4. 'Some problems of 'Partitio numerorum'; II: Proof that any large number is the sum of at most 21 biquadrates', *Mathematische Zeitschrift*, vol. 9 (1921), pp. 14—27.

G. H. Hardy and S. Ramanujan.

1. 'Une formule asymptotique pour le nombre des partitions de n', *Comptes rendus de l'Académie des Sciences*, 2 Jan. 1917.

2. 'Asymptotic formulae in combinatory analysis', *Proceedings of the London Mathematical Society*, ser. 2, vol. 17 (1918), pp. 75—115.

3. 'On the coefficients in the expansions of certain modular functions', *Proceedings of the Royal Society of London* (A), vol. 95 (1918), pp. 144—155.

E. Landau.

1. 'Zur Hardy-Littlewood'schen Lösung des Waringschen Problems', *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen* (1921), pp. 88—92.

L. J. Mordell.

1. 'On the representations of numbers as the sum of an odd number of squares', *Transactions of the Cambridge Philosophical Society*, vol. 22 (1919), pp. 361—372.

A. Ostrowski.

1. 'Bemerkungen zur Hardy-Littlewood'schen Lösung des Waringschen Problems', *Mathematische Zeitschrift*, vol. 9 (1921), pp. 28—34.

S. Ramanujan.

1. 'On certain trigonometrical sums and their applications in the theory of numbers', *Transactions of the Cambridge Philosophical Society*, vol. 22 (1918), pp. 259—276.

N. M. Shah and B. M. Wilson.

1. 'On an empirical formula connected with Goldbach's Theorem', *Proceedings of the Cambridge Philosophical Society*, vol. 19 (1919), pp. 238—244.