USING COMPLETELY IMPLICIT METHODS TO SOLVE THE DIRECT PROBLEM OF NOZZLE THEORY

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An iterative method from the class of completely implicit methods (CIM) is proposed for solving the equation of the velocity potential of elliptical-hyperbolic type. The advantage of the proposed method compared to other CIM schemes is demonstrated in application to mixed perfect gas flows in a nozzle of a given shape. Accuracy issues, rate of convergence, and various techniques of ensuring stability in the hyperbolic region are discussed.

1. INTRODUCTION

Increasing attention is being devoted in recent years in computational aerodynamics to the development of fast and accurate numerical methods for solving the mixed-type full velocity potential equation. Recent numerical methods [1, 2] ensure very fast convergence of the solution with an optimal choice of iteration parameters if certain specific conditions are satisfied. However, in practice, the optimal values of the iteration parameters are not known in advance. As a result, these methods may prove to be inefficient in applications.

When problems for the equation of the velocity potential are solved by difference methods, we usually have to solve systems consisting of a large number of linear algebraic equations with matrices of strictly ordered structure. Using the five-point scheme for approximation of derivatives to the two-dimensional full velocity potential equation, we obtain a system of equations with a five-diagonal matrix. The nine-point scheme leads to a nine-diagonal system of difference equations.

The general stationary problem is representable in matrix form as

\[ A\phi = q \]  \hspace{1cm} (1)

where \( A \) is the strongly sparse matrix of coefficients, \( \phi \) is the vector of unknown values of the velocity potential at the nodes of the grid covering the numerical region, \( q \) is the column vector of unknowns.

The method of successive upper relaxation by lines (SURL) [2, 3] is the most widespread iterative method for solving the full velocity potential equation. However, the implicit variable-direction method (VDM) [3, 4] has faster convergence than SURL. In VDM, the matrix of the system of equations is transformed in each iteration so that a system of equations with a tridiagonal matrix corresponds to each coordinate line. The solution of the modified system involves successively traversing the numerical region "line by line" in all the coordinate directions and inverting the corresponding system of difference equations by a direct method - the sweeping method. This is a largely implicit method, and it substantially increases the rate of convergence compared with SURL. It uses one or several parameters to accelerate convergence, which naturally raises difficulties with the choice of appropriate parameters. Poor choice of these parameters may substantially slow down convergence and even produce instability.

The variable-direction method is currently facing competition from other iterative methods. There are many such methods, and the most successful among them are the so-called approximate-factorized methods (AFM) [5, 12]. Although AFM is faster, it requires special skills from the user. The characteristics of AFM also substantially depend on the choice of the iteration parameters and boundary conditions for the intermediate variable.

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The purpose of this article is to describe two iterative methods which are comparable with AFM by rate of convergence. The idea of a completely implicit method (CIM) for solving systems of difference equations was first proposed in [6]. The author only demonstrated the possibility of solving linear partial differential equations of elliptical type by this method. CIM, like VDM, generates factorized difference schemes. It can be applied with equal success to construct relaxation schemes for solving stationary equations and as a stabilization method.

We focus on the efficiency of CIM and its modifications for solving stationary potential problems of interior aerodynamics. We basically consider numerical solution of equations for the velocity potential in the two-dimensional case. We compare the computational results obtained for the direct problem of two-dimensional Laval nozzles by CIM and its modifications. The possibility of generalizing these methods to the three dimensional case is considered.

2. STATEMENT OF THE PROBLEM

Consider mixed perfect gas flow inside an axisymmetric nozzle. The working region is bounded by the nozzle contour, the input and output sections of the nozzle, and the symmetry axis. The partial differential equation describing two-dimensional inviscid isentropic gas flows in the Cartesian system of coordinates is representable in conservative form as

\[(y\rho\phi_x)_x + (y\rho\phi_y)_y = 0,\]  \hspace{1cm} (2)

where

\[\rho = \left(\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}\left(\phi_x^2 + \phi_y^2\right)^{1/2}\right).\]

Equation (2) is the full velocity potential equation. In this equation, \(\phi\) is the velocity potential, \(\rho\) is the gas density, \(\gamma\) is the specific heat ratio. In the transonic problem, we have to solve Eq. (2) with the boundary condition of impermeability on the contour \(d\phi/dn = 0\) and a given value of the potential \(\phi\) in the output section. Equation (2) is nonlinear, because the density \(\rho\) is a function of the potential derivatives \(\phi_x\) and \(\phi_y\).

Numerical solution of this equation for transonic flow is very difficult, because the equation changes its type from elliptical in the subsonic region to hyperbolic in the supersonic region, and the boundary between these two regions is not known in advance.

For calculation of two-dimensional nozzles of arbitrary configuration, we pass from Cartesian to curvilinear coordinates, mapping the physical flow region onto a rectangular numerical region. In the transformed coordinates \(\xi\) and \(\eta\), Eq. (2) takes the form

\[\frac{\partial}{\partial \xi} \left(y\rho\left(D_1\phi_x - D_2\phi_\eta\right)\right) + \frac{\partial}{\partial \eta} \left(y\rho\left(D_3\phi_\eta - D_4\phi_\xi\right)\right) = 0,\]  \hspace{1cm} (3)

where

\[\rho = \left(\frac{\gamma + 1}{2} - \frac{\gamma - 1}{2}\left(D_1\phi_x^2 + 2D_2\phi_x\phi_\eta + D_3\phi_\eta^2\right)^{1/2}\right).\]

Here

\[D_1 = \frac{x^2 + y^2}{J}, D_2 = \frac{x_\xi x_\eta + y_\xi y_\eta}{J}, D_3 = \frac{x^2 + y^2}{J},\]

are the metric transformation coefficients, and \(J = x_\eta y_\xi - x_\xi y_\eta\) is the Jacobian of the inverse transformation.