THE PROBABILITY FUNCTION OF A GENERALIZED POISSON DISTRIBUTION AND SPECIAL POLYNOMIALS

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The probability function of a generalized Poisson distribution is written in terms of modified Hermite polynomials of 2nd kind in many variables. Their relationship is established with special polynomials previously derived for this purpose. The properties of the polynomials and the probability function are studied.

1. STATEMENT OF PROBABILISTIC PROBLEMS

The generalized Poisson distribution (GPD) is defined as follows [1-3]. Given are a random variable (r.v.) $N$ that follows the Poisson distribution with the parameter $\lambda$, and a collection $\xi_1, \xi_2, \ldots$ of identically distributed r.v.s that are independent of one another and of $N$. The distribution of the r.v. $\xi = \xi_1 + \xi_2 + \ldots + \xi_N$ is GPD. In the class of non-negative integer r.v.s the generating function (g.f.) for $z$ is

$$q(z) = \frac{\lambda}{\lambda + 1} \left( 1 - \lambda z \right), \quad |z| \leq 1,$$

where $q(z)$ is the g.f. of the components $\xi_i$.

The explicit form of the probability function of the distributions (1) is usually defined by various special polynomials [4-9]. For the problems considered in this article, in addition to previously defined polynomials $H_n$ [8, 12, 13] we also use Hermite polynomials in many variables $H_n^*$ [8, 10, 11].

If $\xi_i$ follows the distribution $p_n$ concentrated at $k + 1$ points,

$$p_0 = 1 - \frac{1}{\lambda} \sum_{\nu=1}^{k} \lambda^{\nu}, \quad p_\nu = \lambda^{\nu}, \quad \nu = 1, \ldots, k, \quad p_0 = 0,$$

where $\lambda > 0$, then the GPD is defined by the generating function

$$q(z) = \exp \left\{ \sum_{\nu=1}^{k} \lambda^{\nu} (z^\nu - 1) \right\}.$$

The distribution of the sum $\xi = \eta_1 + \eta_2 + \ldots + \eta_k$ of $k$ so-called multiple Poisson variables, where the generating function of each $\eta_\nu$ is

$$\varphi(z) = \exp \{ \lambda(z^{\nu} - 1) \}, \quad \nu = 1, \ldots, k.$$

is also reducible to the same law. Note that the generating function of the weighted r.v. $\xi = \xi_1 + 2\xi_2 + \ldots + k\xi_k$, where $\xi_\nu$ are independent Poisson r.v.s with parameters $\lambda_\nu > 0$, $\nu = 1, \ldots, k$, is also represented by (2).

Let us formulate a nontrivial problem concerning (2). Suppose that under the given conditions on $\xi_\nu$, $\nu = 1, \ldots, k$, we know that the conditional r.v. $\xi|\xi$ follows the binomial distribution with the parameters $\xi$ and $e$ ($0 < e \leq 1$). It is required to find the distribution of the unconditional r.v. $x$. The g.f. for $\xi$ is (2) with the parameters
\[ \theta_j = e^j \sum_{v=j}^{k} C_j^v (1 - \varepsilon)^v \lambda_v, \quad j = 1, \ldots, k; \]

instead of \( \lambda_v, v = 1, \ldots, k \) [8, 12, 13].

The probability function corresponding to (2) is written as

\[ P_n = \frac{P_0}{n!} H_n(\lambda_1, \ldots, \lambda_k), \quad P_0 = \exp\{- \sum_{v=1}^{k} \lambda_v\}, \quad n = 0, 1, \ldots, \]

\[ H_n(\lambda_1, \ldots, \lambda_k) = n! \sum \prod_{v=1}^{k} \frac{1}{i_v!} \lambda_v^{i_v} \]

(the sum is over integer nonnegative solutions of the equation \( i_1 + 2i_2 + \ldots + ki_k = n \)) the polynomials \( H_n = H_n(\lambda_1, \ldots, \lambda_k) \) are introduced in [6, 8] in explicit form (5) and are defined by the exponential generating function (e.g.f.)

\[ H(t) = \sum_{n=0}^{\infty} H_n(k_1, \ldots, k_k) \frac{t^n}{n!} = \exp\{ \sum_{v=1}^{k} \lambda_v t^v \}. \]

We use the notation \( I_{\mu}^{(\nu)} = \sum_{j=\mu}^{k} j \mu \) so that the summation condition in (5) is written in the form \( I_{\mu}^{(k)} = n \). The superscript \( k \) of \( I_{\mu} \) will be omitted whenever there is no danger of confusion.

2. MODIFIED HERMITE POLYNOMIALS OF MANY VARIABLES

The polynomials \( H_n \) are related with polynomials of degree \( n \) in \( k - 1 \) variables:

\[ H_{e_n}(x_2, \ldots, x_k) = n! \prod_{i=2}^{k} \sum_{i_v=0}^{\infty} \frac{1}{i_v!} \left( \frac{x_v}{i_v} \right)^{i_v} \times \frac{x_2^{n-i_2-\ldots-i_k}}{(n - I_2)!}, \]

\( k = 2, 3, \ldots \) (the superscript \( k \) has been omitted for \( I_2 \)); for \( k = 1 \) by definition we set \( H_{e_n}(x_2, \ldots, x_k) = x_2^n \) and for all \( k \) we set \( H_{e_n}(x_2, \ldots, x_k) = 0 \) for \( n = -1, -2, \ldots \). These polynomials are naturally called modified Hermite polynomials of second type of many variables. The e.g.f. of these polynomials is given by the expression

\[ H^* (t) = H^* (t; x_2, \ldots, x_k) = \exp\{x_2 t + \frac{t^2}{2} + \sum_{v=3}^{k} x_v \frac{t^v}{v} \}. \]

Indeed, substituting in the definition of e.g.f. (7), changing the order of summation \( k - 1 \) times, and using the obvious equalities \( (n = 2, \ldots, k) \)

\[ \{ n = I_2^{(\nu-1)}, \ldots, \infty; i_v = 0, \ldots, \frac{(n - I_2^{(\nu-1)})}{v} \} = \{ i_v = 0, \ldots, \infty; n = I_2^{(\nu)}, \ldots, \infty \} \]

we obtain

\[ H^* (t) = \sum_{n=0}^{\infty} H_{e_n} \frac{t^n}{n!} = \prod_{i=2}^{k} \sum_{i_v=0}^{\infty} \frac{1}{i_v!} \left( \frac{x_v}{i_v} \right)^{i_v} \sum_{n=0}^{\infty} \frac{x_2^{n-i_2-\ldots-i_k}}{(n - I_2)!} \frac{t^n}{n!} \]