A COMPUTATIONAL ALGORITHM FOR THE SECOND TERM OF THE SERIES OF THE RAY METHOD IN AN INHOMOGENEOUS ISOTROPIC ELASTIC MEDIUM

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An algorithm for computing the second term of the series of the ray method in the case of elastic inhomogeneous isotropic media is proposed. The main idea of the approach to the problem can be formulated as follows. Let a central (or support) ray of a ray tube be known. If we introduce the ray-centered coordinates \( s, q_1, q_2 \) in the vicinity of the central ray, then the rays of the ray tube can be described by functions \( q_i = q_i(s, \gamma_1, \gamma_2), \) \( i = 1, 2, \) where \( s \) is the arc length of the central ray and \( \gamma_j, \) \( j = 1, 2, \) are the ray parameters. On the one hand, we show that the integrand of the second term of the series of the ray method can be expressed via the derivatives of the functions \( q_i \) with respect to \( \gamma_j \) of the first, second, and third orders. On the other hand, additional differential equations for the derivatives as functions of \( s \) can be obtained from Euler's equations for the rays. This paper also contains initial conditions for the derivatives in the case of a point source. Thus, we obtain an algorithm involving additional differential equations for the derivatives \( \frac{\partial q_i}{\partial \gamma_j}, \frac{\partial^2 q_i}{\partial \gamma_j \partial \gamma_k}, \frac{\partial^3 q_i}{\partial \gamma_j \partial \gamma_k \partial \gamma_l} \) and the initial conditions for them at the source. The algorithm for calculating the mixed components of a vector of displacement is elaborated in detail. Unfortunately, the Russian version of this paper contains some errors, which have been corrected in the English translation of the paper. Bibliography: 14 titles.

The problem of estimating the contribution of the second term of the series of the ray method to wave fields has stimulated considerable interest in seismic wave propagation (see, for example, the recent publications [1–3]). This, in turn, has focused the attention of researchers on developing relevant computational algorithms for the second term which should be effective and not time consuming. The main difficulty that stands in the way is that of evaluating the derivatives of the first and second order of the geometrical spreading with respect to the ray parameters as functions of a point placed on a given ray. Obviously, if the rays can be found in explicit analytic form, the derivatives will also be available in explicit analytic form and, consequently, the second term of the series of the ray method can be evaluated by means of the known formulas (see [4, 5]). But, for more or less general models of inhomogeneous elastic media, direct numerical computations are required for constructing the rays, which leads to the difficult problem of computing the derivatives of the geometrical spreading with respect to the ray parameters.

An algorithm for computing the mixed component of a vector of displacement was suggested in [6–8]; the calculation of the mixed term, as a part of the second term, only requires a knowledge of the derivatives of the first order of the geometrical spreading. The aim of the present paper is to develop an algorithm for computing the whole second term of the ray series.

We consider an inhomogeneous isotropic elastic medium with Lamé's parameters \( \lambda \) and \( \mu \) and the density \( \rho \) assumed to be smooth functions. We write the ray series in the form

\[
\tilde{U}(x, y, z, t) = \exp \left[ i \omega (t - \tau(x, y, z)) \right] \sum_{k=0}^{\infty} (i \omega)^{-k} \tilde{u}_k(x, y, z),
\]

(0.1)

where \( \tilde{U} \) is a vector of displacement of a wave field, \( \omega \) is the frequency, and \( \tau \) is the eikonal. The principal term of the ray series corresponds to \( k = 0 \) in Eq. (0.1), and for \( k = 1 \) we obtain the second term, which is the subject matter of our consideration.

Further, we consider the case of compressional waves, for which the formulas for \( \tilde{u}_0 \) are as follows (see [4, 5]):

\[
\tilde{u}_0 = \varphi_0 \nabla \tau
\]

\[
\varphi_0 = \psi_0(\gamma_1, \gamma_2) \sqrt{\frac{\alpha}{\rho J}},
\]

(0.2)

where $\nabla \tau$ is the gradient of the eikonal, $\gamma_1, \gamma_2$ are the ray parameters, the function $\psi_0(\gamma_1, \gamma_2)$ describes the initial data and depends upon the source of the wave field; $a = ((\lambda + 2\mu)/\rho)^{1/2}$ is the velocity of compressional waves, and we denote by $J$ the geometrical spreading of the ray tube,

$$J = \left| \frac{D(x,y,z)}{D(\tau, \gamma_1, \gamma_2)} \right|. \quad (0.3)$$

The expression for $\vec{u}_1$ in Eq. (0.1) under consideration can be represented in the form

$$\vec{u}_1 = \varphi_1 \nabla \tau + \vec{u}_1^{(0)}, \quad (0.4)$$

where the vector $\vec{u}_1^{(0)}$ is orthogonal to $\nabla \tau$ and is usually called the mixed component of a wave field. It can be written as follows:

$$\vec{u}_1^{(0)} = -\frac{a^2}{\lambda + \mu} \frac{M(\tau, \varphi_0 \nabla \tau)}{\rho \frac{a^2}{2\rho} \sqrt{\rho} \frac{J}{\rho} \left( -M(\tau, \vec{u}_1^{(0)}) + L(\vec{u}_0), \nabla \tau \right) dt}. \quad (0.5)$$

The formula for $\varphi_1$ in Eq. (0.4) reads

$$\varphi_1 = \sqrt{\frac{a}{\mu}} \left[ \psi_1(\gamma_1, \gamma_2) + \int_0^\tau \frac{a^2}{2\rho} \sqrt{\rho} \frac{J}{a} \left( -M(\tau, \vec{u}_1^{(0)}) + L(\vec{u}_0), \nabla \tau \right) dt \right], \quad (0.6)$$

where the function $\psi_1(\gamma_1, \gamma_2)$ describes the initial data for the ray series (0.1), and $M(\tau, \vec{u})$ and $L(\vec{u})$ are certain differential operators applied to a vector function $\vec{u}$. They are well known in ray theory for elastic media, and exact formulas for them in Cartesian coordinates can be found in [5]. These formulas are rather cumbersome, and so we do not write them here. We note that $M$ contains derivatives of the first order in the direction orthogonal to a given ray, while $L$ involves derivatives of both the first and second order.

Thus, in accordance with Eq. (0.6), we must know these derivatives as functions of the eikonal $\tau$ on the ray in order to evaluate the second term $\vec{u}_1$. The main idea of our approach to the problem can be formulated as follows. Assume that a central ray of a ray tube is known and given in the form $\vec{r} = \vec{r}_0(s)$, where $\vec{r}$ is the radius vector and $s$ is the arc length of the ray. We then introduce the ray-centered coordinates $s, q_1, q_2$ with origin on the ray $\vec{r}_0(s)$ by the formula

$$\vec{r}_M = \vec{r}_0(s) + \sum_{i=1}^2 q_i \vec{e}_i(s), \quad (0.7)$$

where $\vec{r}_M$ is the radius vector of a point $M$ in the vicinity of the ray $\vec{r}_0(s)$, and $\vec{e}_1(s), \vec{e}_2(s)$ are two orthogonal unit vectors, both orthogonal to the ray $\vec{r}_0(s)$ at every point of the ray; for details, see [9, 10]. Then the rays of the ray tube can be described by functions

$$q_j = q_j(s, \gamma_1, \gamma_2), \quad j = 1, 2, \quad (0.8)$$

which satisfy Euler's equations for the corresponding Fermat integral. It is obvious that the geometrical spreading (0.3) and the other functions in Eqs. (0.2)-(0.6) can be expressed via functions (0.8), and the derivatives in the direction orthogonal to the central ray give rise to the corresponding derivatives of functions (0.8) with respect to the ray parameters $\gamma_1$ and $\gamma_2$. On the other hand, starting with the Cauchy problem for a ray field and differentiating it with respect to the ray parameters $\gamma_1, \gamma_2$, we obtain a set of Cauchy problems for the derivatives of (0.8) with respect to $\gamma_1, \gamma_2$. Thus, finally, we arrive at an algorithm for computing the second term $\vec{u}_1$ of the ray series (0.1) on a given central ray.

It should be mentioned that the method described above has been used for computation of the geometrical spreading in [10, 11]. A similar approach to computing the mixed component $\vec{u}_1^{(0)}$ in Eq. (0.5) was suggested in [6-8].