ON COMPUTER-ALGEBRA PROCEDURES THAT CHECK FOR COMMON EIGENVECTORS OR INVARIANT SUBSPACES

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We give a survey of the theoretical facts that make it possible to establish the existence of common eigenvectors or two-dimensional invariant subspaces for given matrices $A$ and $B$ with rational elements. We describe procedures in the language MAPLE constructed on the basis of this theory. We discuss methods of generating test matrices for these procedures and the results of numerical experiments carried out for them.

Bibliography 20 titles.

Let $A$ and $B$ be given $n \times n$ matrices with integer or rational elements. How can we verify the existence or nonexistence of common eigenvectors for $A$ and $B$ over the complex field $\mathbb{C}$? The obvious answer—compute separately the eigenvectors of the two matrices and then compare them—is not applicable, not only because there is usually too much freedom in the choice of eigenvectors when there are multiple eigenvalues. Even if $A$ and $B$ are matrices with a simple spectrum having a common characteristic direction, the computations will yield only an approximation of that direction, different for $A$ and $B$. In the optimal case these approximations will be very similar, but only a special and nontrivial analysis can conclude positively from this similarity that there is a common eigenvector.

As it happens, the question of a common eigenvector can be solved using a finite algorithm that depends rationally on the elements of the given matrices. In other words, a question about the field $\mathbb{C}$ can be given an exact answer using computations within the rational field $\mathbb{Q}$.

A survey of the theoretical facts that form the basis of this possibility, and also some facts connected with related questions (the existence of common invariant subspaces of dimension larger than one for the matrices $A$ and $B$ or the simultaneous reducibility of these matrices to triangular, block-triangular, or block-diagonal form), is given in Section 1 of this article. In the second section we list some computational procedures constructed on the basis of this theory and realized in the framework of the popular computer-algebra system MAPLE. The results of numerical experiments with these procedures are discussed in Section 4. In these experiments we use several families of test matrices whose properties are of independent interest. A brief description of these matrices and theories explaining their properties will be found in Section 3.

1. On common eigenvectors and simultaneous reducibility. A necessary and sufficient condition for the $n \times n$ matrices $A$ and $B$ to have a common eigenvector was found comparatively recently, to be specific in an article of Shemesh [1] published in 1984. We set

$$N = \bigcap_{k, l=1}^{n-1} \ker [A^k, B^l].$$

The symbol $[C, D]$ denotes the commutator of the matrices $C$ and $D$: $[C, D] = CD - DC$.

Theorem 1 (Shemesh' criterion). The matrices $A$ and $B$ have a common eigenvector if and only if

$$N \neq \{0\}. \quad (2)$$

To give condition (2) a constructive form, we introduce the matrix

$$K = \sum_{k, l=1}^{n-1} ([A^k, B^l]^*) [A^k, B^l]. \quad (3)$$

Translated from *Metody Matematicheskogo Modelirovaniya*, 1998, pp. 5–23.
Theorem 2. The matrices $A$ and $B$ of order $n$ have a common eigenvector if and only if the $n \times n$ matrix $K$ is singular.

Another constructive modification of the Shemesh criterion uses the matrix $L$ of dimensions $(n - 1)^2 \times n$, constructed as follows:

$$L = \begin{bmatrix}
[A \cdot B] \\
[A \cdot B^2] \\
\vdots \\
[A \cdot B^{n-1}]
\end{bmatrix}$$

Theorem 3. The matrices $A$ and $B$ of order $n$ have a common eigenvector if and only if $\text{rank } L < n$.

Remark. If it is known that the degree $n_1$ (resp. $n_2$) of the minimal polynomial of $A$ (resp. $B$) is strictly less than $n$, it is possible without changing the subspace $N$ to reduce the description (1) for it to the form

$$N = \bigcap_{1 \leq k < m} \ker [A^k, B^k]$$

This makes it possible to reduce the number of terms in the formation of the matrix $K$ to $(n_1 - 1)(n_2 - 1)$. The number of block rows in the matrix (4) can be reduced similarly.

By modifying Shemesh' criterion we obtain methods of solving different variants of the common eigenvalue problem. Suppose, for example, that along with the matrices $A$ and $B$ we are also given a polynomial in two variables $p(x, y)$ and required to determine whether $A$ and $B$ have a common eigenvector $x$,

$$Ax = \lambda x, \quad Bx = \mu x, \quad x \neq 0,$$

such that the following condition holds for the corresponding eigenvalues $\lambda$ and $\mu$:

$$p(\lambda, \mu) = 0.$$ 

Theorem 4. The problem (6)-(7) has a solution if and only if

$$N \cap \ker p(A, B) \neq 0.$$ 

Here $p(A, B)$ is the matrix obtained from $p(x, y)$ through formal replacement of $x$ by $A$ and $y$ by $B$.

Corollary. A necessary and sufficient condition for the matrices $A$ and $B$ to have a common characteristic pair $\lambda$, $x$ (that is, an eigenvalue and an eigenvector corresponding to it) is that

$$N \cap \ker (A - B)^r \neq \{0\}.$$ 

Condition (8) can be given a constructive form in a way similar to what was done for the basic Shemesh criterion (2). Thus, it is necessary to add a block row to the matrix (4), namely the matrix $F = p(A, B)$. In the sum on the right-hand side of formula (3) it is necessary to add the term $F \ast F$.

We note another interesting result contained in [1]. Let $A$ be, as before, an $n \times n$-matrix, and $B$ a rectangular $m \times n$ matrix. It is necessary to determine whether $A$ has an eigenvector $x$ belonging to the subspace $\ker B$:

$$Ax = \lambda x, \quad Bx = 0, \quad x \neq 0.$$ 

Theorem 5. The problem (9) has a solution if and only if