Axiomatics of Newtonian Cosmology.

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Summary. - Four axioms are assumed which, essentially, state the existence of a frequency function, a qualitative law of gravitation, the conservation of mass (or probability), Newton's law of motion and the local homogeneity of the Universe (§ 2). Theorems I and II yield a complete survey over all the universes which satisfy these axioms and an additional regularity hypothesis (§ 3). Some kinematical and thermodynamical features of these universes are discussed in § 4.

§ 1 - Introduction.

Cosmology based on Newtonian dynamics encounters the difficulty that Newton's law of gravitation and the «World Postulate» («Cosmological Principle») are incompatible. According to this postulate the mass-density is supposed to be independent of the position vector \(x\) (in the euclidean 3-space). Therefore the gravitational potential, \(V(x, t)\) (where \(t\) denotes the time variable), should be independent of \(x\) also. The potential \(V(x, t)\) and the mass-density \(\varphi(x | t)\), however, are related by Poisson's equation

\[
\Delta V(x, t) = 4\pi G \varphi(x | t)
\]

(\(\Delta\) is the Laplace operator and \(G\) is the constant of gravitation) — which implies that \(V(x, t)\) is a non-constant function of \(x\). (Cf. [5], [8], [9]).

This inconsistency could be removed, for example, by the following modification of Poisson's law:

\[
\Delta V(x, t) = 4\pi G \left[ \varphi(x | t) - \bar{\varphi}(t) \right]
\]

where \(\bar{\varphi}(t)\) is the «mean density of the universe at the time \(t\) ». It admits universes of which both the mass-density \(\varphi(x | t)\) and the gravitational potential \(V(x, t)\) are independent of the position \(x\). In such a universe the mass-elements move according to Galileo's law of inertia.

It is, however, by no means necessary to specify the modified law of gravitation; instead, it is sufficient to postulate that the gravitational force is uniquely determined by the mass-distribution of the universe. Then a mass-density which is independent of the position \(x\) implies a gravitational potential independent of \(x\). It is the purpose of this note to discuss the consequences of such a qualitative law of gravitation and a strictly local world postulate.
§ 2. - Axioms.

The term «universe» is taken as an undefined primitive notion. Its properties will formally be described by a number of definitions and axioms. The following spaces are assigned to it:

(i) the «real space», \( X \), i.e., the set of all «position vectors» \( x \). «Vector» means «column vector». The number of dimensions, \( n \), of \( X \) is not specified since nearly all results are independent of the particular value of \( n \);

(ii) the associate «velocity space» \( U \), i.e., the set of all «velocity vectors» \( u \); both \( X \) and \( U \) are \( n \)-dimensional vector spaces;

(iii) the «phase space» \( \Gamma = U \times X \);

(iv) the «time-axis» \( T \), i.e., the set of all real numbers \( t \);

(v) \( T' \), an open interval of the time axis which contains the time zero, \( t = 0 \);

(vi) the cartesian products \( \Gamma \times T \) and \( \Gamma \times T' \).

To any universe a «frequency function» \( f(u, x | t) \) is assigned which is defined on \( \Gamma \times T' \). One intuitive interpretation of it reads: let \( M \) be any measurable bounded subset of \( \Gamma \); then

\[
\int_{M} f(u, x | t) du dx
\]

is the mass contained in \( M \) at the time \( t \). Thus a universe is pictured by a continuous material substratum which fills the phase space and has the mass-density \( f(u, x | t) \) in \( \Gamma \times T' \).

Axiom I. The frequency function, \( f(u, x | t) \), of a universe is defined and non-negative on \( \Gamma \times T' \), and is positive for at least one value of \( (u, x) \) and \( t = 0 \). It has continuous derivatives of first order with respect to the components of \( (u, x, t) \) everywhere in \( \Gamma \times T' \). The moments of the orders 0, 1 and 2,

\[
\int_{\Gamma} f(u, x | t) \, du ,
\int_{\Gamma} uf(u, x | t) \, du ,
\int_{\Gamma} u_i u_j f(u, x | t) \, du ,
\]

\( i \leq t, j \leq n \).