On the Discreteness of the Spectrum associated with certain Differential Equations.

Memoria di E. C. Titchmarsh (a Oxford).

Sunto. - I teoremi di Weyl e di Friedrichs, sulla discretezza degli spettri relativi alle seguenti equazioni (1.1) ed (1.2), vengon qui stabiliti con un metodo più semplice, estendibile al caso in cui le variabili siano in numero qualsiasi.

1. Consider first the problem of the eigenvalues of the ordinary differential equation

\[ \frac{d^2 \Phi}{dx^2} + \lambda - q(x) \Phi = 0, \]

where \( x \) ranges over \((-\infty, \infty)\), and \( q(x) \to \infty \) as \( x \to \infty \) and as \( x \to -\infty \). It was proved by Weyl that in this case the spectrum, viz. the set of eigenvalues, is discrete, i.e. that it consists of isolated points. His argument depends on considerations of the number of zeros of the eigenfunctions (1).

The corresponding theorem for more variables than one seems to have been first proved by Friedrichs (2), by quite a different method. In the case of two variables, the theorem is that the spectrum associated with the partial differential equation

\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \lambda - q(x, y) \Phi = 0, \]

where \((x, y)\) ranges over the whole plane, is discrete if \( q(x, y) \to \infty \) as \( r = \sqrt{x^2 + y^2} \to \infty \).

The object of the present paper is to give simpler proofs of these theorems.

I first give a new proof for the case of one variable, and then solve the two-variable problem in a similar way. The method could obviously be extended to more than two variables.

---


2. I begin by recalling the situation in the one-variable case. It is sufficient to consider the half-line \( x \geq 0 \), with the boundary condition \( \Phi = 0 \) at \( x = 0 \). In this case the eigenvalues are the poles of the Green's function \( G(x, \xi, \lambda) \), which, considered as a function of \( x \), satisfies (1.1) except at \( x = \xi \), and \( G(0, \xi, \lambda) = 0 \), while \( G(x, \xi, \lambda) \) is \( L^2 \) over \((0, \infty)\). At \( x = \xi \), \( G \) is continuous, but \( \partial G/\partial x \) has a discontinuity, and increases by 1.

If \( \theta(x, \lambda) \), \( \Phi(x, \lambda) \) are the solutions of (1.1) such that

\[
\begin{align*}
\theta(0, \lambda) &= 1, & \theta'(0, \lambda) &= 0, \\
\Phi(0, \lambda) &= 1, & \Phi'(0, \lambda) &= -1,
\end{align*}
\]

then there is a function of \( \lambda \), \( m(\lambda) \), such that \( \psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\Phi(x, \lambda) \) is \( L^2 \); and then

\[
G(x, \xi, \lambda) = -\psi(x, \lambda)\Phi(\xi, \lambda), \quad (\xi \leq x),
\]

\[
= -\Phi(x, \lambda)\psi(\xi, \lambda), \quad (\xi > x).
\]

In the case of a finite interval \((0, b)\), with eigenfunctions vanishing at each end of the interval, the corresponding Green's function is

\[
G_b(x, \xi, \lambda) = \psi_b(x, \lambda)\Phi(\xi, \lambda), \quad (0 \leq \xi \leq x \leq b),
\]

\[
= -\Phi(x, \lambda)\psi_b(\xi, \lambda), \quad (0 \leq x \leq \xi \leq b),
\]

where

\[
\psi_b(x, \lambda) = \theta(x, \lambda) + M(\lambda)\Phi(x, \lambda),
\]

\( M(\lambda) \) being defined by

\[
\theta(b, \lambda) + M(\lambda)\Phi(b, \lambda) = 0.
\]

The functions \( M(\lambda) \) and \( M(\lambda) \) are analytic functions of \( \lambda \), regular except possibly on the real axis, and \( M(\lambda) \to m(\lambda) \) as \( b \to \infty \), for every non-real \( \lambda \).

The function \( M(\lambda) \) satisfies an inequality of the form (2)

\[
| M(\lambda) | < A + \frac{B}{|v|},
\]

where \( \lambda = u + iv \), and \( A \) and \( B \) are independent of \( b \). Hence \( M(\lambda) \) also satisfies the same inequality, and, by Vitali's convergence theorem (4), \( M(\lambda) \to m(\lambda) \) uniformly in any finite region entirely in the upper (or lower) half-plane.

The only singularities of \( G_b(x, \xi, \lambda) \) are simple poles at certain points \( \lambda_m, b \) on the real axis, the eigenvalues for the finite interval \((0, b)\). Now each eigenvalue associated with a given \( q(x) \) and a certain interval is less than the corresponding eigenvalue associated with the same \( q(x) \) and an interval included in the former one (4). Hence, as \( b \) increases, each eigenvalue \( \lambda_m, b \)

---

(1) See § 2, 2 of Eigenfunction Expansions.


(3) R. Courant and D. Hilbert, Methoden der math Physik, I, Kap. VI, § 2, Satz. 3.