On characterization of probability distributions
by means of independent statistics.

IGNAC¥ I. KOTLARSK[, Oklahoma State University (U.S.A.) (*)

Summary. - The probability density functions \( f_k(x_k) = A_k \cdot x_k^{-\beta_k-1} e^{-a_k(x_k)} \) of independent random variables \( x_1, x_2, \ldots, x_n \), are characterized by independence of two functions of them.

TAMHANKAR [3] gave a characterization of the normal distribution symmetric about the origin by means of independent statistics. KOTLARSKI [2] characterized the gamma distribution in a similar way. In this paper the same method is used to characterize other distributions, where the proceeding two cases are particular ones.

Let \( x_0, x_1, \ldots, x_n \) \((n \geq 1)\) be independent real random variables (r.v.-s) having probability density functions (p.d.f.-s) \( f_k \) satisfying the following conditions:

\[
(A_1) \quad f_k(x) > 0 \text{ for } x \in \mathcal{A}_k, \text{ where } \mathcal{A}_k = (-\infty, 0) \text{ or } (0, +\infty) \text{ or } (-\infty, 0) \cup (0, +\infty); f_k(x) = 0 \text{ for } x \notin \mathcal{A}_k;
\]

if \( \mathcal{A}_k = (-\infty, 0) \cup (0, +\infty) \) then \( f_k \) is even;

\( f_k \) is continuous on \( \mathcal{A}_k; \)

\[
(A_2) \quad \text{there exist real numbers } p_k \text{ such that the following limits exist}
\]

\[
(1) \quad \lim_{x \to 0} \frac{f_k(x)}{|x|^{\beta_k-1}} = A_k > 0.
\]

Denote the following transformation

\[
(2) \quad \begin{cases}
\quad x_0 = y \cdot h_0(y_1, \ldots, y_n) \\
\quad x_1 = y \cdot h_1(y_1, \ldots, y_n) \\
\quad \vdots \\
\quad x_n = y \cdot h_n(y_1, \ldots, y_n)
\end{cases}
\]

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for $x_k \in \mathcal{A}_k$, $k = 0, 1, \ldots, n$, $y \in \mathcal{B} = (0, \infty)$, $(y_1, \ldots, y_n) \in \mathcal{B}$, where $\mathcal{B}$ is a set in the $n$-dimensional space. Assume that the functions $h_k$ on $\mathcal{B}$ are taken in such a way, that there is possible the change of variables in $(n + 1)$-dimensional integrals, particularly that the JACOBIAN of (2) exists and does not vanish for $y \in \mathcal{B}_0$, $(y_1, \ldots, y_n) \in \mathcal{B}$.

Assume further that there exist functions $\Psi(y)$, $y \in \mathcal{B}_0$ transforming $\mathcal{B}_0$ onto $\mathcal{C} = (0, \infty)$ and $\Psi_k(x)$, $x \in \mathcal{A}_k$, $k = 0, 1, \ldots, n$ transforming $\mathcal{A}_k$ onto $\mathcal{C}$, satisfying the following conditions:

**\text{(B}_1\text{)}** $\Psi$ is strictly increasing on $\mathcal{B}_0$,

- if $(-\infty, 0) \subset \mathcal{A}_k$ then $\Psi_k$ is decreasing on it,
- if $(0, \infty) \subset \mathcal{A}_k$ then $\Psi_k$ is increasing on it,
- if $\mathcal{A}_k = (-\infty, 0) \cup (0, \infty)$ then $\Psi_k$ is even;

**\text{(B}_2\text{)}** there exist limits

$$\lim_{y \to 0} \Psi(y) = 0, \quad \lim_{x \to 0} \Psi_k(x) = 0$$

**\text{(B}_3\text{)}** $\Psi(y) = \Psi_0(x_0) + \Psi_1(x_1) + \ldots + \Psi_n(x_n)$, $x_k \in \mathcal{A}_k$, $k = 0, 1, \ldots, n$, $y \in \mathcal{B}_0$;

**\text{(B}_4\text{)}** the functions

$$f_i(x) = \begin{cases} A_k |x|^p \text{e}^{-x \Psi_k(x)} & x \in \mathcal{A}_k \\ 0 & x \notin \mathcal{A}_k \end{cases}$$

where $p_k$ and $A_k$ are given in (\text{A}_2), are p.d.f.-s.

**Theorem.** - The necessary and sufficient condition for the r.v.-s $x_k$ ($k = 0, 1, \ldots, n$) to be distributed according to the p.d.f.-s (4) is that

**\text{(C)}** $y$ and $(y_1, \ldots, y_n)$ are independent.

**Proof.** - The $(n + 1)$-dimensional r.v. $(y, y_1, \ldots, y_n)$ has its p.d.f. $g(y, y_1, \ldots, y_n)$ given on $\mathcal{B}^* = \mathcal{B}_0 \times \mathcal{B}$ by the formula

$$g(y, y_1, \ldots, y_n) = f_0(x_0) \cdot f_1(x_1) \ldots f_n(x_n) \cdot |J|$$