On analytic primals.

Memoria di Jerzy Herszberg (London, W. C. 1, Gran Bretagna)

Summary. - Certain standard forms of equations of primals with singularities of a given type are investigated. Also the relation between the geometric and analytic case is given and some analytic invariants are found. The application of these methods to the resolution of singularities is indicated.

1. In a paper on resolution of singularities [3] B. Segre used certain forms of local equations of $d$-folds lying on non-singular $(d+1)$-folds. He derived, by means of these methods, some results about the geometric properties of the variety. Some properties of analytic primals were later on discussed by him in [4].

In the extensive paper [3] only a small part of it was devoted to a detailed discussion of the relation between the geometric and analytic methods. In this paper we show, by means of an example, that the method of choosing local equations may change in some cases the geometric properties of the variety at the point in question. Furthermore we derive a set of conditions under which we can apply the methods of analytic transformations, thereby simplifying, and not altering, the geometric situation. We also obtain some new results in this connection.

2. In this paragraph we give an example of a possible behaviour, under analytic transformations, of a surface $F$ in $S_4$ with a double curve $C$. For simplicity we use non-homogeneous coordinates $x, y, z$.

Let $F$ be the surface whose equation is

$$(x^2 + 2xy + y^3) f_1 (x, y, z) + 2(x^2 + 2xy + y^3) z f_2 (x, y, z) + x^2 f_3 (x, y, z) = 0,$$

where $f_2(0, 0, 0) \neq 0$ and

$$[f_2(0, 0, 0)]^2 - f_2(0, 0, 0) f_1(0, 0, 0) f_3(0, 0, 0) \neq 0,$$

but $f_1, f_2, f_3$ are otherwise general.

This surface $F$ has a double point at the origin and on $F$ there is a
The curve \( C \) has a double point at \( O \) and the use of dilatations [3] is restricted to bases which are non-singular. Thus to resolve the singularity of \( F \) at \( O \) (or, strictly speaking, the singularity of \( F \) through \( O \)) we first have to apply a dilatation with \( O \) as base. We obtain thereby the proper transform of \( S \), which is a non-singular threefold \( M \), and the proper transform \( F_1 \) of \( F \) on it. On \( F_1 \) there is a double curve \( C_1 \), which is the proper transform of \( C \) and a double curve \( E_1 \) corresponding to the point \( O \). To resolve the singularities of \( F_1 \) we have to apply, at least, two more dilatations, one with \( C_1 \) as base and one with \( E_1 \) (or, strictly speaking, with its proper transform) as base, in total, at least three dilatations.

Now in the problem of resolution of singularities one often deals with a sequence of consecutive points and with the behaviour of a given variety at these points. To facilitate the analysis one often allows the use of local coordinates which are analytic transforms of \( x, y, z \), regular at the point in question. Returning now to the surface \( F \), suppose we are interested in the behaviour of \( F \) at \( O \) and we allow the following analytic transformation:

\[
\begin{align*}
x_1 &= x + 2y - \frac{1}{2}y^2 - \frac{1}{8}y^3 - \frac{1}{16}y^4 - \frac{5}{128}y^5 - \ldots = x + y + \sqrt{1-y}, \\
y_1 &= x + \frac{1}{2}y^2 + \frac{1}{8}y^3 + \frac{1}{16}y^4 + \frac{5}{128}y^5 + \ldots = x + y - y\sqrt{1-y}, \\
z_1 &= z.
\end{align*}
\]

This transformation is certainly regular at \( O \). Then \( x_1y_1 = x^2 + 2xy + y^2 \) and the «local equation» of \( F \) may be written in the form

\[x_1^2y_1^2F_4(x_1, y_1, z_1) + 2x_1y_1z_1F_3(x_1, y_1, z_1) + z_1^2F_2(x_1, y_1, z_1) = 0,
\]

where \( F_i(x_1, y_1, z_1) \) is a power series, convergent for small

\[
|x_1|, |y_1|, |z_1| \quad \text{for} \quad i = 1, 2, 3, \text{and} \quad F_2(0, 0, 0) \neq 0.
\]

Here we have two double «lines» through \( O \), given by \( x_1 = x_1 = 0 \) and \( y_1 = z_1 = 0 \) and to resolve the singularity we first apply the local dilatation

\[
x_1 = x_2, \quad y_1 = y_2, \quad z_1 = z_2 y_2,
\]