Critical Point Theory and the Number of Solutions of a Nonlinear Dirichlet Problem (*).

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Introduction and Summary.

The motivation for this paper stems from the following result:

**Theorem D.** - Let $D$ be a bounded domain in $\mathbb{R}^n$ whose boundary $\partial D$ is of class $C^{2+\alpha}$ for some $\alpha \in (0, 1)$. Let $\Delta$ denote the Laplacian and the sequence of eigenvalues of the boundary value problem

$$
(\Delta u)(x) + \lambda u(x) = 0 \quad x \in D
$$

$$
u(x) = 0 \quad x \in \partial D,
$$

with each $\lambda_n$ occurring in the sequence as often as its multiplicity. If $g \in C^1(\mathbb{R}, \mathbb{R})$, and there exist constants $\gamma$ and $\gamma'$ and an integer $N$ such that

$$
\lambda_N < \gamma < g'(t) < \gamma' < \lambda_{N+1}
$$

for all $t \in \mathbb{R}$, then for any $p \in C^1(\overline{D})$ there exists a unique solution of the boundary value problem

$$(P) \quad \Delta u(x) + g(u(x)) = p(x) \quad x \in D
$$

$$
u(x) = 0 \quad x \in \partial D.
$$

This result was essentially given by C. L. DOLPH in [11]. Although the result is not explicit in [11], it follows immediately from results concerning nonlinear integral equations of the Hammerstein type via use of the Green's function for the boundary value problem $(\Delta u)(x) = f(x), \ x \in D; \ u(x) = 0, \ x \in \partial D$.

For different derivations which depend on implicit function theoretic arguments we refer the reader to the papers [9] and [16]. Generalizations, which give conditions for existence only, can be found in [13] and [14].

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AMBROSETTI and PRODI studied the boundary value problem \( (P) \) under the assumption that the range of \( g' \) contains an eigenvalue. Specifically, in [2], they showed that if \( g(0) = 0, g''(t) > 0 \) for all \( t \), and \( \lim_{t \to \pm \infty} g'(t) = g'(\pm \infty) \) with
\[
0 < g'(\infty) < \lambda_1 < g'(\infty) < \lambda_2,
\]
then \( (P) \) has either zero, one, or two solutions. More precisely, they showed that in the Banach space \( C^0(D) \) there exists a \( C^2 \) closed manifold \( M \) whose complement consists of two components \( U_0 \) and \( U_2 \) such that \( (P) \) has no solution for \( p \in U_0 \), \( (P) \) has two solutions for \( p \in U_2 \), and \( (P) \) has one solution for \( p \in M \).

In this paper we also consider the boundary value problem under the assumption that the range of \( g' \) contains an eigenvalue. We shall show that slight alterations of the conditions of Theorem D imply nonuniqueness of the solutions of \( (P) \) for suitably restricted \( p(x) \). With stronger assumptions on \( g \) we can give the exact number of solutions. Specifically, we will prove the following results:

**Theorem A.** – Let \( D \) and \( p \) satisfy the same smoothness conditions as in Theorem D. Assume \( g(0) = 0, g' \in C^0(\infty, \infty) \), and \( g' \) is bounded. If there exist an integer \( N \) and numbers \( \gamma \) and \( \gamma' \) such that \( \gamma < \gamma' < \lambda_{N+1} \) with \( g'(t) > \gamma' \) for all \( t \in (-\infty, \infty) \) and
\[
(\ast) \quad -\infty < \inf_{t \in \mathbb{R}} \left[ \int_0^t g(s) ds - \frac{\gamma s^2}{2} \right],
\]
and if
\[
(\ast\ast) \quad g'(0) < \lambda_N,
\]
then the homogeneous problem
\[
(P_0) \quad (Au)(x) + g(u(x)) = 0 \quad x \in D
\]
\[
u(x) = 0 \quad x \in \partial D
\]
has at least two solutions; in particular, there exists a nontrivial solution of \( (P_0) \). If, in addition to \( (\ast\ast) \), we assume that
\[
(\ast\ast\ast) \quad g'(0) \neq \lambda_j \quad \text{for all } j,
\]
then, if the \( L^2(D) \) norm of \( p(x) \) is sufficiently small, the nonhomogeneous problem \( (P) \) has at least three solutions.

**Theorem B.** – Let \( p \) and \( D \) satisfy the same smoothness conditions as in Theorem D. Assume that \( g(0) = 0, g' \in C^0(\mathbb{R}, \mathbb{R}) \), and that \( t g''(t) > 0 \) almost everywhere. If \( \lim_{t \to \infty} g'(t) = g'(\infty) \) and \( \lim_{t \to -\infty} g'(t) = g'(-\infty) \) are finite and there exists an integer \( N \) such that\n\[
(a) \ \lambda_{N-1} < g'(0) < \lambda_N,
\]