which completes the proof of the theorem.

LITERATURE CITED


SOME EXTREMAL PROPERTIES OF POSITIVE TRIGONOMETRIC POLYNOMIALS

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For \( n = 8 \) an upper bound is given for the functional

\[
V_n = \inf_{t_n} \frac{a_1 + a_2 + \ldots + a_n}{(\sqrt{a_1} - \sqrt{a_0})^2},
\]

which is defined on the class of even, nonnegative, trigonometric polynomials

\[ t_n(\varphi) = \sum_{k=0}^{n} a_k \cos k\varphi, \]

such that \( a_k \geq 0 \) for \( k = 0, \ldots, n \), \( a_k > a_0 \), \( V_n \leq 54.54 \cdot 10^6 \).

In problems on the distribution of the prime numbers or zeros of the Riemann zeta function an important role is played by the functional

\[
V_n = \inf_{t_n \in P_n} \frac{a_0}{(\sqrt{a_1} - \sqrt{a_0})^2}, \quad \inf_{t_n \in P_n} \frac{a_1 + \ldots + a_n}{(\sqrt{a_1} - \sqrt{a_0})^2},
\]

defined on the class \( P_n \) of even trigonometric polynomials

\[
t_n(\varphi) = a_0 + a_1 \cos \varphi + \ldots + a_n \cos n\varphi,
\]

which satisfy the following conditions:

a) \( t_n(\varphi) \geq 0 \), for all \( \varphi \);

b) \( a_k \geq 0 \), for \( k = 0, \ldots, n \);

c) \( a_1 > a_0 \).

By using the polynomial

\[
t_2(\varphi) = 2(1 + \cos \varphi)^2 = 3 + 4 \cos \varphi + \cos 2\varphi,
\]

Vallee Poussin showed in [1] that the Riemann zeta function \( \zeta(\sigma + it) \) has no complex zeros in the domain

\[
\sigma \geq 1 - 1/(R \ln t) \quad (t \geq T).
\]
for some (sufficiently large) $R$ and $T = 12$, and he then produced the following estimate on the error in the asymptotic formula for the number of prime numbers $\pi(x)$ not exceeding $x$:

$$r(x) = \pi(x) - \int_1^x \frac{du}{\ln u} = O(xe^{-K\sqrt{x}}), \quad K = 0.186.$$  \hfill (3)

In [2] and §§65, 79, and 81 of [3] Landau showed that for an arbitrary $\varepsilon > 0$ in (2) we can set

$$R = \frac{V}{2} + \varepsilon, \quad T = T(\varepsilon),$$  \hfill (4)

where $V = \lim_{n \to \infty} V_n$.

In [4] Stechkin showed that for an arbitrary $\varepsilon > 0$ in (2) we can set

$$R = \frac{3\sqrt{V_4}}{10} V + \varepsilon, \quad T = T(\varepsilon).$$  \hfill (5)

From (4) and (5) it follows that in order to estimate $R$ we need suitable upper bounds on the $V_n$, which are obtained by constructing nearly extremal polynomials $t_\varepsilon \in P_n$.

In this regard, the extremal problem (1) has been investigated by many authors, and upper as well as lower bounds have been obtained for $V_n$ ($n = 2, 3, 4$) and $V$ in [3] and [5-8]. Special attention was given to the case $n = 4$. Rosser and Schoenfeld gave an example in [5]

$$t_4(q) = 2 (1 + \cos q)^2 (3 + 10 \cos q)^2,$$

which showed that $V_4 < 35.03$...

By setting

$$t_4(q) = (0.28 + \cos q)^2 (0.91 + \cos q)^2,$$

Stechkin showed in [4] that

$$V_4 < 34.9.$$  

Calculations by Hollenbeck (see [6, p. 249]) showed that for the polynomial

$$t_4(q) = \alpha + \cos q)^2 (b - \cos q)^2$$

with $\alpha = 0.9126$ and $b = 0.2766$ we have $V_4 < 34.8993$, and in (2), by virtue of (5), we can set $R = 9.645908801$.

In [7] French obtained the lower bound

$$V \geq 32.51 \ldots,$$

and in [8] Stechkin showed that

$$V_4 \geq 34.35.$$  

In this work we introduce a polynomial $t_\varepsilon(q)$, for which $V_4 \geq 34.54461566$ and $R = 9.54789695$, thereby improving the value of $R$ given in [6] by 0.098. For constructing this polynomial the problem of minimizing (1) was solved on an IBM computer. In order to satisfy condition a) we used the representation of an even, nonnegative, trigonometric polynomial

$$t_n(q) = \sum_{k=0}^{n} a_k \cos kq$$

in the form (see [9])

$$t_n(q) = \left| \sum_{k=0}^{n} c_k e^{ikq} \right|^2,$$  \hfill (6)

where the $c_v (v = 0, \ldots, n)$ are real numbers. Here the coefficients $a_k$ are expressed in terms of the $c_v (v = 0, \ldots, n)$ by the formulas

$$a_0 = \sum_{k=0}^{n} c_k^2, \quad a_k = 2 \sum_{v=0}^{n-k} c_v c_{k+v} \quad (k = 1, \ldots, n).$$  \hfill (7)

Thus we have solved the problem of finding

$$\min_{c_0, \ldots, c_n} V_n (a_0, \ldots, a_n)$$

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