Conditionally Asymptotically Invariant Sets and Perturbed Systems.

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Summary. — In this paper, by introducing the concepts of conditionally asymptotically invariant (CAI) sets and the extreme stability of a set relative to another set, smooth Lyapunov functions are constructed and the preservation of stability behavior of CAI sets under constantly acting perturbations are studied.

1. Introduction.

Recently, using the notion of stability in variation and employing the nonlinear variation of constants formula, some interesting perturbation results of invariant and asymptotically invariant sets have been discussed [1], [3], [6]. Furthermore, introducing the concept of a conditionally invariant set, its stability properties have also been investigated in [2], [4], [7]. We, in this paper, plan to fuse these ideas so as to unify and generalize the earlier results. This leads to the new concept, namely, conditionally asymptotically invariant sets, whose stability behavior implies much weaker notions of stability of motion that might be of practical interest. In this general set up, the nonlinear variation of constants formula and the concept of stability in variation that are used in earlier work, are not indeed applicable. This dictates employing a suitable new notion which we call the extreme stability of a set relative to another set, because of which, construction of smooth Lyapunov functions is possible. It turns out that this new concept is weaker than the known definitions of extreme stability and stability in variation. Appropriate remarks are added in the paper to classify the situations.

2. We consider the differential system

\[ x' = f(t, x), \quad x(t_0) = x_0, \]

where \( f \in C[R^+ \times \bar{S}(B, \varrho), R^n] \). Here \( R^+ \) denotes the non-negative real line, \( R^n \) the euclidian space, \( \bar{S}(B, \varrho) = \{ x \in R^n : d(x, B) < \varrho \} \), \( d(x, B) = \inf_{b \in B} \| x - b \| \), \( \| \cdot \| \) any convenient norm, \( B \) a subset in \( R^n \) and \( C[R^+ \times \bar{S}(B, \varrho), R^n] \) the class of continuous functions from \( R^+ \times \bar{S}(B, \varrho) \) to \( R^n \). Let \( A \) and \( B \) be any two subsets of \( R^n \) such that \( A \subset B \). We need the following definitions.

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DEFINITION 2.1.

i) \( \mathcal{L} = \{ \lambda \in \mathcal{C}[\mathbb{R}^+, \mathbb{R}^+]: \lambda(t) \) is decreasing in \( t \) and \( \lim_{t \to \infty} \lambda(t) = 0 \}; \)

ii) \( \mathcal{K} = \{ a \in \mathcal{C}([0, \infty), \mathbb{R}^+]: a(0) = 0 \) and \( a(r) \) is increasing in \( r \}; \)

iii) \( \mathcal{D} = \{ b \in \mathcal{C}[\mathbb{R}^+ \times [0, \infty), \mathbb{R}^+]: b(t, r) \) is decreasing in \( t \) for each \( r \), increasing in \( r \) for each \( t \) and \( \lim_{t \to \infty} b(t, r) = 0 \}. \)

DEFINITION 2.2. – The set \( B \) is said to be conditionally asymptotically invariant (CAI) relative to the set \( A \) and the differential system (2.1) if there exists a function \( \lambda \in \mathcal{L} \) such that \( x_0 \in A \) implies

\[
d(x(t, t_0, x_0), B) \leq \lambda(t_0), \quad t > t_0,
\]

\( x(t, t_0, x_0) \) being any solution of (2.1).

DEFINITION 2.3. – The CAI set \( B \) relative to the set \( A \) and the differential system (2.1) is said to be

i) uniformly stable if there exist functions \( a \in \mathcal{K} \) and \( \lambda \in \mathcal{L} \) such that

\[
d(x(t, t_0, x_0), B) \leq a\left(d(x_0, A)\right) + \lambda(t_0), \quad t > t_0;
\]

ii) uniformly asymptotically stable if there exist functions \( \beta \in \mathcal{K}, \sigma \in \mathcal{L} \) and \( H \in \mathcal{C}[\mathbb{R}^+ \times [0, \infty), \mathbb{R}^+] \) such that \( \lim_{t \to \infty} H(t, t_0) = 0 \) provided \( t_0 > r > 0 \) for some \( r \) and

\[
d(x(t, t_0, x_0), B) \leq \beta\left(d(x_0, A)\right)\sigma(t - t_0) + (Ht, t_0), \quad t > t_0.
\]

REMARK 2.1. – If, in the definition 2.2., \( B = A \) then \( A \) is said to be asymptotically self invariant (ASI). If, on the otherhand, \( \lambda(t) \equiv 0 \), \( B \) is said to be conditionally invariant with respect to \( A \). Clearly when both \( B = A \) and \( \lambda(t) \equiv 0 \), the set \( A \) is self invariant in the usual sense.

In light of the foregoing remark, the stability definition 2.3 should be interpreted accordingly in the respective cases. See [2], [5], [6] for clarity of the remark.

Corresponding to the given differential system (2.1), we shall consider the scalar differential equation

\[
u' = g(t, \nu), \quad \nu(t_0) = \nu_0 > 0,
\]

where \( g \in \mathcal{C}[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}] \). We shall assume that the set \( \nu = 0 \) is asymptotically self invariant relative to (2.5).

We shall state below a general result which is itself new and which gives sufficient conditions for the set \( B \) to be CAI relative to a set \( A \) and also stability criteria of such a set with respect to the given differential system (2.1).