On a Horizontal Conformal Killing Tensor of Degree $p$ in a Sasakian Space.

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Summary. - We deal with a horizontal conformal Killing tensor of degree $p$ in a Sasakian space. After some preparations we prove that a horizontal conformal Killing tensor of odd degree is necessarily Killing. Moreover, we consider horizontal conformal Killing tensor of even degree. The form of the associated tensor is determined completely and a decomposition theorem is proved. Then we give the examples of a conformal Killing tensor of even degree and a special Killing tensor of odd degree with constant $i$.

Let $M$ be an $n$-dimensional Riemannian space whose metric tensor is given by $g_{ab} (a, b, \ldots, r, s, \ldots = 1, 2, \ldots, n)$. We call a skew-symmetric tensor $u_{ab}$ a conformal Killing tensor of degree 2 if it satisfies the equation

$$\nabla_a u_{bc} + \nabla_b u_{ac} = 2 \theta_c g_{ab} - \theta_a g_{bc} - \theta_b g_{ac},$$

where $\nabla$ denotes the operator of covariant derivative with respect to $g_{ab}$. Then we have $\theta_c = \nabla^a u_{ac}/(n - 1)$ for the tensor $u_{ab}$. We call $\theta_c$ the associated vector (1) of $u_{ab}$.

Recently, the author [6] has studied a conformal Killing tensor of degree 2 in a Sasakian space and obtained the followings:

**Theorem A.** - In a Sasakian space ($n > 3$), any conformal Killing tensor $u_{ab}$ is uniquely decomposed into the form:

$$u_{ab} = w_{ab} + q_{ab},$$

where $w_{ab}$ is Killing and $q_{ab}$ is a closed conformal Killing tensor. In this case $q_{ab}$ is the form

$$q_{ab} = -\nabla_c \theta_c,$$

where $\theta_c$ is the associated vector of $u_{ab}$.

**Theorem B.** - Let $M$ be a complete simply connected Sasakian space ($n > 3$) admitting a conformal Killing tensor $u_{ab}$ whose associated vector is $\theta_c$. If the inner

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(1) We adapt the identification between vector fields and 1-forms by virtue of Riemannian metric.
product $\langle \theta, \theta \rangle$ or $\langle \eta, \theta \rangle$ is not constant, where the vector $\eta$ is a Sasakian structure, then $M$ is isometric with a unit sphere.

In a Riemannian space, T. Kashiwada [1] has defined a conformal Killing tensor of degree $p \geq 2$ and generalized some results about a conformal Killing tensor of degree 2 to the case of degree $p \geq 2$.

The purpose of this paper deals with a horizontal conformal Killing tensor of degree $p$ in a Sasakian space. After some preparations we prove in § 4 that a horizontal conformal Killing tensor of odd degree is necessarily Killing. In § 5 we shall consider horizontal conformal Killing tensors of even degree. The form of the associated tensor is determined completely and a decomposition theorem is proved (cf. Theorem 5.1 and 5.3).

1. - Tensors.

In a Riemannian space $M$ we call a $p$-form $w$ with coefficient $w_{a_{1}..a_{p}}$ a Killing tensor of degree $p$ if it satisfies

$$\nabla_{b}w_{a_{1}..a_{p}} + \nabla_{a_{1}}w_{b..a_{p}} = 0.$$

If a Killing tensor $w$ satisfies

$$\nabla_{a_{1}}\nabla_{a_{2}}w_{a_{3}..a_{p}} + \alpha \left( g_{a_{1}a_{2}}w_{a_{3}..a_{p}} + \sum_{i=3}^{p} (-1)^{i} g_{a_{1}a_{i}}w_{a_{3}..a_{i-1}a_{i+1}..a_{p}} \right) = 0,$$

where $\alpha$ is constant and $\delta_{i}$ means that $a_{i}$ is omitted, then it is called a special Killing tensor of degree $p$ with constant $\alpha$ [5].

Next we shall remember a conformal Killing tensor of degree $p$. In $M$ we call a $p$-form $u$ with coefficient $u_{a_{1}..a_{p}}$ a conformal Killing tensor of degree $p$, if there exists a $(p-1)$-form $\theta$ with coefficient $\theta_{a_{1}..a_{p}}$ such that

$$\nabla_{b_{1}}u_{a_{1}..a_{p}} + \nabla_{a_{1}}u_{b_{1}..a_{p}} = 2\theta_{a_{1}..a_{p}}g_{b_{1}b_{2}} - \sum_{i=2}^{p} (-1)^{i} \left( \theta_{a_{1}..a_{i}a_{i+1}..a_{p}} + \theta_{b_{1}..a_{i}a_{i+1}..a_{p}}g_{b_{1}b_{2}} \right).$$

This form $\theta$ is called the associated tensor of $u$. For a conformal Killing tensor $u$ of degree $p$, the following identities are known:

$$\nabla^{r}u_{a_{1}..a_{p}} = (n-p+1)\theta_{a_{1}..a_{p}},$$

$$\begin{align*}
(p-1)I_{b_{1}..b_{p}} + \sum_{r=2}^{p} I_{a_{1}..b_{1}..a_{p}} \\
= -\frac{1}{n-p} \left[ (p-1)R_{b_{1}..b_{p}}u_{a_{1}..a_{p}} + (p-2) \sum_{r=2}^{p} R_{b_{1}..b_{r}}u_{a_{1}..a_{r-1}a_{r+1}..a_{p}} \\
+ \sum_{i=2}^{p} \sum_{a_{i}<a_{i+1}} R_{a_{i}..a_{i+1}}u_{a_{1}..a_{i-1}a_{i+2}..a_{p}} - \sum_{2 \leq i < j}^{p} R_{a_{i}..a_{j}}u_{a_{1}..a_{i-1}a_{i+2}..a_{p}} \right].
\end{align*}$$