On \(T\)-accretive Operators.

BRUCE D. CALVERT (Roma) (*) (**)

**Sunto.** – Sono date condizioni su alcuni operatori differenziali che assicurano la \(T\)-accretività negli spazi \(L^p\).

**Introduction.**

The aim of this discussion is to present concrete differential operators which are the generators of nonlinear order preserving semigroups in spaces of real valued \(L^p\) functions on a Riemannian manifold. Such operators have been studied in the writer's paper [8]. The simplest central property of such operators is that they should be \(T\)-accretive or even \(m\)-\(T\)-accretive, defined below.

In Section 1, definitions are given, after which it is shown that operators of order \(m > 2\) are not \(T\)-accretive in any \(L^p\) space, and are not even accretive in \(L^p\) if \(p\) is large.

In Section 2, we prove that second order operators arising from elliptic boundary value systems in variational form are \(m\)-\(T\)-accretive under simple conditions on the coefficients. In the exposition we deal with linear and also nonlinear operators. We deal with a compact manifold, but for clarity we consider subsets of \(R^n\) and postpone consideration of manifold until the appendix.

In Appendix 1, we consider \(\Omega\) a manifold. One reason for working on manifolds is that in a manifold there may be an empty boundary, but for a bounded set \(\Omega \subset R^n\) this is not so. Moreover, the boundary operators act on a nontrivial manifold even when \(\Omega \subset R^n\).

In Appendix 2, we compare and contrast the properties of \(T\)-accretivity with the weak maximum principle, and the property \(a(u^-, u^+) = 0\) for \(u\) in \(H^1\), and \(T\)-monotonicity, i.e. \(a(u, u^+) > 0\) for \(u\) in \(H^1\).

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1. – Higher order operators.

We now define \(T\)-accretivity, first abstractly then in particular in \(L^p\) spaces. (The lattice notation appears for example in YOSIDA [22]).

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A function \( L: X \rightarrow P(X) \) from a Banach lattice \( X \) to subsets of \( X \) is said to be \( T \)-accretive (\( T \) stands for truncation) iff for \( \lambda > 0 \) and \( x, y \) in the image \( R(I + \lambda L) \),

\[
\| [(I + \lambda L)^{-1} x - (I + \lambda L)^{-1} y]^+ \| < \| (x - y)^+ \|.
\]

Equivalently, given \( x \) in \( Lu \), \( y \) in \( Lv \), there exists \( f \) such that \( f>0 \) in \( X^* \), \( \|f\| < 1 \), \( (f, u - v) = \|(u - v)^+\| \), and \( (f, x - y) > 0 \). This enables us to give the concrete definitions below. \( L \) is hypermaximal \( T \)-accretive, or briefly \( m \)-\( T \)-accretive, if also \( R(I + \lambda L) = X \) for some (hence all) \( \lambda > 0 \).

1. Let \( \Omega \) be a closed bounded subset of \( R^n \) with \( C^\infty \) boundary \( \partial \Omega \) and nonempty interior \( \Omega^o \).

In the space \( L^p(\Omega) \), \( p \in (1, \infty) \), \( T \)-accretivity is equivalent, in the case that \( L \) is single valued, to

\[
\int_{u \geq v} (Lu - Lv)(u - v)^{p-1} ds > 0.
\]

In the general case the previous formula holds in the sense that for \( x \) in \( Lu \), \( y \) in \( Lv \),

\[
\int (x - y)(s)[(u - v)(s)]^{p-1} ds > 0
\]

where the integral is taken over the set \( \{s: u(s) > v(s)\} \).

\( L \) is \( T \)-accretive in \( L^1(\Omega) \) iff for \( x \) in \( Lu \), \( y \) in \( Lv \) there exists a measurable function \( f \) with \( 0 < f < 1 \), \( \int |u - v| = 0 \), such that if \( \psi \) is the characteristic function of the set \( \{s: u(s) > v(s)\} \),

\[
\int (x - y)(f + \psi) ds > 0.
\]

\( L \) is \( T \)-accretive in \( L^2(\Omega) \) iff for \( x \) in \( Lu \), \( y \) in \( Lv \), there is a finitely additive absolutely continuous positive set function \( \varphi \) which has total variation 1, satisfies

\[
\int (u - v)(s)\varphi(ds) = \text{ess. sup} (u - v)^+ \]

and is such that

\[
\int (x - y)(s)\varphi(ds) > 0.
\]

These results follow from the characterisation of the dual of \( L^p \) as in YoSida [22, Chapter IV.9].

Stampacchia [21, page 280] remarks that we are obliged to consider second order operators if we use comparison theorems. The reason why the arguments treating operators of second order do not work is because in particular \( H^m_0(\Omega^o) \) is not a sublattice of \( L^1 \) for \( m > 2 \). For it is enough to take \( u \) in \( C^\infty_0 (\Omega^o) \), \( u(\bar{x}) = x_1 \) for \( x \) in the