On Two Conjectures of P. Chowla and S. Chowla Concerning Continued Fractions.

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To Professor Beniamino Segre on his 70-th birthday.

Summary. - The alternating sum of the partial quotients in the primitive period of a continued fraction expansion of \( \sqrt{D} \) is determined mod 2 and mod 3.

Let \( D \) be a non-square positive integer and set

\[
\sqrt{D} = b_0 + \frac{1}{b_1} + \cdots + \frac{1}{b_k} = [b_0, b_1, \ldots, b_k],
\]

where the bar denotes the primitive period,

\[
\Sigma_D = b_k - b_{k-1} + \cdots + (-1)^{k-1} b_1.
\]

P. Chowla and S. Chowla [1] have made among others the following conjectures:

- if \( D \equiv 3 \mod 4, 3 \nmid D \) then \( \Sigma_D \equiv 0 \mod 3 \),
- if \( p, q \) are primes; \( p \equiv 3 \mod 4, q \equiv 5 \mod 8 \) then

\[
(-1)^{\frac{p}{2}} = \left( \frac{2}{q} \right).
\]

The aim of this paper is to prove two theorems which generalize the above conjectures (Note that \( D \equiv 3 \mod 4 \) implies \( k \equiv 0 \mod 2 \)).

Theorem 1. - If \( k \) is even, \( 3 \nmid D \) then \( \Sigma_D \equiv 0 \mod 3 \).

Theorem 2. - \( \Sigma_D \equiv \nu \mod 2 \), where \( u, v \) is the least non-trivial solution of \( U^2 - DV^2 = 1 \). Moreover, let \( 2 \nmid D, k(D) \) be the squarefree kernel of \( D \) and \( C \) any divisor of \( 2k(D) \) different from \( 1 \) and \( -k(D) \) such that \( U^2 - k(D)V^2 = C \) is soluble. If either \( C \equiv 1 \mod 2 \) or each prime factor of \( D \) divides \( k(D) \) then \( \Sigma_D \equiv C + 1 \mod 2 \).

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Remark. — If both conditions given in Theorem 2 for odd $D$ are violated the conclusion may fail, e.g. for $D = 147 = 3 \cdot 7^2$ we have $\Sigma_D = 16 \equiv 0 \mod 2$, although $u^2 - 3v^2 = -2$ is soluble.

Corollary 1. — If $D \equiv 0, 3, 7 \mod 8$ then $\Sigma_D = 0 \mod 2$.

Corollary 2. — If $p$ is a prime, $p \equiv 3 \mod 4$; $\alpha$ is odd then $\Sigma_\pi = 1 \mod 2$.

Corollary 3. — If $p, q$ are primes, $p \equiv 3 \mod 4$, $q \equiv 5 \mod 8$, $\alpha, \beta$ are odd then

$$(-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} = \left(\frac{p}{q}\right).$$

Proof is based on several known facts from the classical theory of continued fractions which we quote below in form of lemmata from the book of Perron [4]. First however we must recall Perron's notation. For a given regular continued fraction $[b_0, b_1, ..., b_n]$ the Muir symbol $K\left(\begin{array}{c}1, 1, ..., 1 \\
b_0, b_1, b_2, ..., b_n\end{array}\right)$ denotes its numerator $A_\alpha$ computed from the formulæ

$$A_{-1} = 1, \quad A_0 = b_0, \quad A_\alpha = b_\alpha A_{\alpha-1} + A_{\alpha-2}.$$

Then we set

$$A_{\nu, \lambda} = K\left(\begin{array}{c}1, ..., 1 \\
b_\nu, b_{\nu+1}, ..., b_{\nu+\lambda}\end{array}\right), \quad A_{\nu, 0} = A_\nu,$$

$$B_{\nu, \lambda} = K\left(\begin{array}{c}1, ..., 1 \\
b_{\nu+1}, b_{\nu+2}, ..., b_{\nu+\lambda}\end{array}\right), \quad B_{\nu, 0} = B_\nu.$$

Lemma 1. — The following formulæ hold

1. $$A_{\nu, \lambda} = b_{\nu, \lambda} A_{\nu-1, \lambda} + A_{\nu-2, \lambda},$$
2. $$B_{\nu, \lambda} = A_{\nu-1, \lambda+1},$$
3. $$A_{\nu, \lambda} = b_\nu A_{\nu-1, \lambda+1} + B_{\nu-1, \lambda+1},$$
4. $$A_{\nu, \lambda} B_{\nu-1, \lambda} - A_{\nu-1, \lambda} B_{\nu, \lambda} = (-1)^{\nu-1}.$$

Proof. — (1) follows directly from the definition of $A_{\nu, \lambda}$. For the remaining formulæ see [4] p. 15, formulæ (25) and (29); p. 17, formula (35).