The existence of non-degenerate functions on a compact differentiable $m$-manifold $M$.

by MARSTON MORSE (a Princeton, N. J.)

To Giovanni Sansone on his 70th birth day.

Summary. - Let $M$ be a compact differentiable $m$-manifold of class $C^m$ in $E_n$, $n = 2m + 1$. Let $x = (x_1, ..., x_n)$ represent a point in $E_n$. The union of the direction $e$ on the direction sphere $S_{n-1}$ in $E_n$ such that the scalar product $e \cdot x$ defines a non-degenerate function on $M$ is an open subset of $S_{n-1}$ whose complement $\omega$ has a Lebesgue measure zero on $S_{n-1}$. When $M$ is non-compact $\omega$ can be everywhere dense on $S_{n-1}$, but still has Lebesgue measure zero.

§ 1. Introduction. - Let $M$ be a compact $m$-manifold ($m > 0$) with a differentiable structure of class $C^\infty$. Cf. Ref [1] for definitions. Let

$$(1.1) \quad (u) \rightarrow A(u): U \rightarrow X$$

be a (1.1) mapping of an open subset $U$ of a euclidean $m$-space of coordinates $(u)$ onto an open subset $X$ of $M$ such that $A$ is compatible with the $C^\infty$-differentiable structure of $M$ given with $M$. Let $f$ be a real-valued function on $M$ of class $C^\infty$, with values $f(p)$ for $p \in M$. For $(u) \in U$ set

$$(1.2) \quad (f(p)|p = A(u)) = \varphi(u).$$

The critical points of $f$ in $X$ are by definition the images $A(u)$ of the critical points $(u)$ of $\varphi(u)$. Any such critical point is termed non-degenerate if and only if the Hessian of $\varphi$ at the critical point $(u)$ does not vanish. The condition that a point $p \in M$ be a critical point of $f$, and, if critical, be degenerate or non-degenerate is clearly independent of the choice of the mapping $A$ used to represent a neighborhood of $p$, among mappings $A$ that are compatible with the given differential structure of $M$. A function, all of whose critical points are non degenerate, is termed non-degenerate. It is clear that the critical points of a non-degenerate function are isolated and hence finite in number.

This paper is concerned with the existence on $M$ of non-degenerate functions $f$ of class $C^\infty$.

In accordance with a theorem of H. Whitney, Ref [3], the $m$-manifold $M$ admits a regular (1-1) mapping of class $C^\infty$ onto some compact differentiable manifold $M'$ of class $C^\infty$ in any euclidean space $E_n$ for which $n \geq 2m + 1$. 
In this theorem it is understood that the differential structure of $M'$ is such that in some neighborhood $R$ relative to $M'$ of each point of $M'$ a suitable subset of $m$ of the euclidean coordinates $(x_1, \ldots, x_n)$ in $E_n$ will serve as «local coordinates» in an admissible representation of $R$ of class $C^\infty$. By virtue of Whitney's theorem, no generality will be lost in our search for non-degenerate functions on $M$ if we assume that $M$ is embedded in the sense of Whitney in a euclidean space $E_n$ for which $n > m$. We shall make this assumption.

Let $S_{n-1}$ be the unit sphere in $E_n$ with center at the origin. A point $(c)$ on $S_{n-1}$ will be represented as a vector

$$c = (c_1, \ldots, c_n).$$

In terms of the rectangular coordinates $(x)$ of a point $p \in E_n$ we introduce the coordinate vector

$$x = (x_1, \ldots, x_n).$$

If $p$ is a point on $M$ let $g(p)$ be the corresponding coordinate vector $x$. For each fixed point $c$ on $S_{n-1}$ the scalar product

$$c \cdot x = c_1 x_1 + \ldots + c_n x_n$$

enables us to define a function $G_c$ on $M$ with values

$$(1.3) \quad G_c(p) = (c \cdot x) \mid (x = g(p)). \quad (p \in M).$$

We shall prove the following theorem:

**Theorem 1.1.** The union of the points $c$ on $S_{n-1}$ such that the function $G_c$ defined on $M$ by (1.3) is non-degenerate is an open set on $S_{n-1}$ whose complement $\omega$ has a Lebesgue measure zero on $S_{n-1}$.

**Coordinate systems $X$ and $Y$.** A coordinate system in which a point $p \in E_n$ is represented by a set of coordinates $(x_1, \ldots, x_n)$ or $(y_1, \ldots, y_n)$ will be termed a coordinate system $X$ or $Y$, respectively.

§ 2. - The $n$-manifold $\Lambda$ of elements normal to $M$.

We shall make use of different rectangular coordinate systems in $E$. These systems shall, however, have a common origin of coordinates. We regard the points $p$ of $E_n$, and in particular the points of $M$ as fixed, represented by different coordinate vectors in different coordinate systems. If $x$ and $y$ are the coordinate vectors of the same point $p \in E_n$ in systems $X$ and $Y$, then $y$ shall equal $T x$, where $T$ is an orthogonal transformation. We say that the systems $X$ and $Y$ are orthogonally related under $T$. We shall similarly refer to certain nonnull vectors $u$ in $E_n$ including vectors normal to $M$ at some point $p$ of $M$. Such vectors $u$, like the points $p \in M$, shall be regarded as fixed when the coordinate axes are rotated. If the vector $u$ is represented by a coordinate vector $a$ in the coordinate system $X$, and if $X$ and $Y$ are