On the structure of some abstract differential problems. - I.

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Summary. - Some differential type problems are formulated abstractly in terms of diagrams of Hilbert spaces and related Schwartz kernels and their structure is studied.

1. Introduction. In recent years boundary value and CAUCHY problems have been treated with outstanding success by the methods of functional analysis (see for example [3; 24; 29] where further references can also be found). Various operational features of general existence-uniqueness theory have been apprehended and in particular a lot of work has devoted to the study of CAUCHY problems with operator coefficients (see for example [3; 4; 5; 10; 11; 12; 13; 17; 21; 23; 27; 28; 29; 33; 41; 43; 44]. This leads then to the natural step of throwing away the remaining differential operator in order to see what operational properties of it are needed for an existence-uniqueness theory involving linked operators of more general nature [4; 5; 9; 14; 16]. On the other hand, using the idea of a space of abstract boundary conditions, one can formulate an abstract idea of what a boundary value problem is (see [3; 8; 14; 15; 18; 25; 45]). In particular one natural procedure is to start with a pair $(S_0, S_0')$ of formally adjoint, one-to-one closed, densely defined, operators in say a HILBERT space $H$ (thus for all $u \in D(S_0), v \in D(S_0')$, $(S_0 u, v) = (u, S_0' v)$). Then $S_0 \subset (S_0')^* = S_1$ and one tries to find a closed densely defined operator $\tilde{S}$ with $S_0 \subset \tilde{S} \subset S_1$ such that $\tilde{S}$ maps one-to-one onto $H$ and has a continuous inverse (see [3]). This situation is indicated diagrammatically in [14; 15] and can be exploited also in the abstract CAUCHY problems indicated and in the more general linked operator problem (see [14]).

In this paper we will apply methods of L. SCHWARTZ ([35; 38; 39; 40]) in order to formulate some of the abstract operator problems referred to above completely and solely in terms of diagrams of HILBERT spaces and their related SCHWARTZ kernels. The original operators are "expendable" for many questions and it is triples and pairs of HILBERT spaces which determine, up to isomorphisms, the "structure" of such operator problems. This kind of development also has functorial aspects, some of which are indicated

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but we shall minimize the "dependence" on categories. Finally index and homotopy arguments will be seen to have a natural setting in this context and this will be developed in part II.

We remark that [40] appeared after the present article was accepted for publication and hence we continue to follow Schwartz's earlier notation [38]. There is some intersection here with [40], indicated in the text (cf. theorems 3 and 4), but since either the proof or the point of view is somewhat different in these instances we have left it alone and included such cases for completeness.

One idea suggested by this sort of procedure is that for linear problems the study of operators or equations, in certain of its aspects, can be replaced by the study of Hilbert subspaces of a fixed Hilbert space, and their Schwartz kernels. For example, one looks at diagrams involving $N(S), D(S),$ and $R(S)$ (with suitable topologies on these spaces) and finds canonical maps of these spaces, determined, independently of $S$, by Schwartz kernels. We recall also that in quantum mechanics there occurs the possibility of studying the Hilbert spaces associated with various elementary particles, and their Schwartz kernels, instead of asking questions about the relativistic wave equations related to these particles (see [35; 39]). Of course one does not completely recover $S$ from $D(S)$ with the graph topology and the Schwartz kernel but for "structural" features $S$ is not needed (a possible "definition" of structure for our purposes).

2. Let $H \subset D'(R^n)$ (see [36; 37] for information about distributions) be a Hilbert space and $A$ an unbounded, densely defined, closed operator in $H$; let $H_A = D(A)$ with the graph topology and scalar product. Then $H_A \subset H$ with a finer topology and we consider the Schwartz anti-kernel $L_A$ (relative to $H$) corresponding to $H_A$ (see [38; 39; 40]). Thus $L_A \in \mathfrak{F}(H', H)$ is defined as the composition $L_A = j' \cdot \theta \cdot \gamma: H' \rightarrow H$ as indicated by

\[
(2.1)
\]

where $j: H_A \rightarrow H$ is the canonical injection and $\theta$ is the canonical anti-isomorphism $H'_{A} \rightarrow H_{A}$ defined by $(y, \theta x')_{A} = \langle y, x' \rangle_{A}, y \in H_A, x' \in H'_{A}$ (here $\langle , \rangle_{A}$ denotes the duality bracket between $H_A$ and $H'_{A}$ and $\mathfrak{F}$ denotes continuous anti-linear maps). One can say that $L_A h' \in H_A \subset H, h' \in H'$, is characterized by the equation

\[
(2.2)
\]

where $\langle , \rangle$ denotes the duality between $H$ and $H'$. To see this note that $(h, L_A h')_A = \langle h, \theta h' \rangle_{A} = \langle h, j' h' \rangle_{A} = \langle j h, h' \rangle = \langle h, h' \rangle$. Thus (cf. [38; 40],

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(2.3)
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where $\langle , \rangle$ denotes the duality between $H$ and $H'$. To see this note that $(h, L_A h')_A = \langle h, \theta h' \rangle_{A} = \langle h, j' h' \rangle_{A} = \langle j h, h' \rangle = \langle h, h' \rangle$. Thus (cf. [38; 40],

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