The singular Cauchy problem
for a non-linear hyperbolic equation

Seymour Singer (Indiana - U.S.A.)

Summary. - The author demonstrates the existence of a smooth solution to a singular initial value problem for a quasilinear hyperbolic equation in two independent variables. The problem is transformed into an equivalent system of integral equations for which a solution is obtained by invoking Schauder's fixed point theorem.

1. - Introduction

In this paper we shall establish the solvability of a Cauchy problem for the second order quasilinear hyperbolic equation

\[
 r(x, y)^2 \frac{\partial^2 u}{\partial y^2} - u_{xx} - u_{yy} + f(x, y, u, u_x, u_y) = 0
\]

subject to the initial conditions

\[
 u(x, 0) = 0, \ u_x(x, 0) = \varphi(x)
\]

prescribed on a bounded segment \( I = [a, b] \) of the \( x \)-axis. \( \beta \) and \( \gamma \) are non-negative real constants at least one of which is positive. The coefficient \( r(x, y) \) is nonvanishing in a neighborhood of \( I \) in the upper half plane. Assuming a certain growth condition on the partial derivative \( f_u \) and a condition connecting the exponents \( \beta, \gamma \) with the initial normal derivative \( \varphi(x) \), we prove the existence of a smooth solution to the Cauchy problem (1.1), (1.2) in a neighborhood of the initial line lying in the upper half plane.

We say \( U(x, y) \) is of class \( \text{Lip}(x, y; M) \) on a domain \( D \) if \( |U(x_1, y) - U(x_2, y)| \leq M|x_1 - x_2| \) for all points \((x_1, y)\) in \( D \), \( i = 1, 2 \). Similarly, \( U(x, y) \) is of class \( \text{Lip}(x, y; M) \) on \( D \) if \( |U(x_1, y) - U(x_2, y)| \leq M(|x_1 - x_2| + |y_1 - y_2|) \) for all points \((x_1, y)\) of \( D \). If \( w \) is a continuous numerical function defined on a bounded domain \( D \), \( \|w\| \) denotes the uniform norm on \( \bar{D} \) i.e., \( \|w\| = \max |w(x, y)| \) for all \((x, y)\in \bar{D}\).

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We shall assume the initial normal derivative \( \varphi(x) \) is of class \( C^3 \) on \( I \), that \( |\varphi|, |\varphi'|, |\varphi''|, |\varphi'''| \) are all \( \leq \|\varphi\| \), and \( \varphi'' \in \text{Lip}(x; B) \) for some positive constant \( B > \|\varphi\| \).

We suppose there exist real constants \( m, n, m', n' \) satisfying

\[
(1.3) \quad m \leq \varphi(x) \leq n, \quad m' \leq \varphi'(x) \leq n'
\]

for all \( x \in I \). Then equation (1.1) is hyperbolic when \( y > 0 \) for every twice-differential solution \( U \) provided

\[
(1.4) \quad m > 0 \text{ if } \beta > 0 \text{ or } m' > 0 \text{ if } \gamma > 0
\]

The problem (1.1), (1.2) is singular since the equation is of parabolic type on the initial segment \( I \).

The coefficient \( r(x, y) \) occurring in (1.1) is assumed positive and bounded away from zero i.e., there exists a number \( \rho > 0 \) such that \( r(x, y) \geq \rho \) for \( x \in I, y \geq 0 \). Furthermore, we assume \( r(x, y) \) is continuous, has two continuous derivatives with respect to \( x \) and one continuous derivative with respect to \( y \). There exists a constant \( \sigma > 0 \) such that \( |r_x|, |r_y|, |r_{xx}|, |r_{yx}| \leq 0 \) and both \( r_{xx} \) and \( r_{yx} \) are of class \( \text{Lip}(x; \sigma) \) when \( x \in I, y \geq 0 \).

Let the curve \( \Gamma_1 \) be the solution of the initial value problem

\[
\frac{dx}{dy} = \sigma(By)^{\beta+\gamma} \\
x = a \text{ when } y = 0.
\]

Let the curve \( \Gamma_2 \) be the solution of the initial value problem

\[
\frac{dx}{dy} = -\sigma(By)^{\beta+\gamma} \\
x = b \text{ when } y = 0.
\]

The arcs \( I, \Gamma_1, \Gamma_2 \) enclose a bounded domain \( D \) contained within the characteristic triangle lying over the interval \( I \). Let \( Y \) denote the maximum ordinate of points in \( D \). For each \( \delta > 0 \) let \( D_\delta \) denote the subset of points \((x, y)\) in \( D \) for which \( 0 < y < \delta \).

Recalling the bounds on \( \varphi, \varphi' \) we select \( a_0 < m \). If \( \beta > 0 \) we may assume \( a_0 > 0 \). Select \( a'_0 < m' \). If \( \gamma > 0 \) we may assume \( a'_0 > 0 \). We choose constants \( A_0 > n, A'_0 > n' \). We also assume \( B \geq A_0, B \geq A'_0 \).

We shall suppose the function \( f(x, y, u, p, q) \) is twice differentiable with respect to \( x, u, p, q \) on the region \( R \) consisting of all points \((x, y, u, p, q)\) such