Extremal barriers on cones with Phragmèn-Lindelöf theorems and other applications

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Summary. - We give a detailed analysis of supersolutions of form $r^2f(0)$ for the class $\mathcal{L}_\alpha$ of uniformly elliptic operators in nondivergence form with ellipticity constant $\alpha$. Fundamental to the analysis is the extremal super'solution $r^2F_{\lambda,\alpha}(\theta)$ for each real $\lambda$, which is itself a solution of a certain extremal elliptic equation and which remains positive on a cone of wider aperture than any other supersolution of this form. These supersolutions are employed as "barriers" to yield Phragmèn-Lindelöf theorems ($\lambda > 0$) for unbounded domains contained in a cone, and Hölder continuity ($\lambda > 0$) and removable singularity ($\lambda < 0$) results at boundary points with an exterior cone property. In each case the growth condition $0(r^2)$ involved, depending on the aperture of the cone, is best possible. Similar results carry through for operators with singular lower order terms and possibly positive zero order coefficient.

In this paper we consider a detailed analysis of supersolutions of form $r^2f(0)$ for the class $\mathcal{L}_\alpha$ of second order uniformly elliptic operators in nondivergence form

$$ Lu = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} $$

with ellipticity constant $\alpha$. Here $0 < \alpha \leq 1$, $(a_{ij}(x))$ is symmetric and has its eigenvalues in $[\alpha, 1]$, $n$ is $\geq 2$, and absolutely no continuity of the coefficients is assumed. Here also $r = |x|$, $\theta$ is the polar angle arccos $|x_n/|x||$, $\lambda$ is any real constant, and

$$ r^2f(0) \text{ is always } C^2 \text{ on } \mathbb{R}^n - \{ \text{closed negative } x_n \text{ axis} \}. $$

Fundamental to the analysis are the extremal supersolutions, solutions of $M_\alpha(r^2f(\theta)) = 0$ where $M_\alpha$ is the maximizing operator introduced by C. Pucci, [14] and [16]. These supersolutions are applied as "barriers at $\infty$" ($\lambda > 0$) for PHRAGMÈN-LINDELÖF theorems on cones, as "barriers" ($\lambda > 0$) for results on continuity at the boundary, as "singular barriers" ($\lambda < 0$) for results on removable singularities at boundary points, and as pathological examples. The extremal supersolution, for fixed $\lambda$, remains positive on a cone of wider aperture than any other supersolution for $\mathcal{L}_\alpha$ of form $r^2f(0)$. The first zeros

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of the extremal supersolutions therefore completely characterize the relationship, for fixed $\omega$, between the aperture $\Psi$ and the possible Hölder exponents $\lambda$ for barriers on $T_\Psi$, the closed right circular cone of this aperture.

This paper is a continuation of a previous paper [8] in which existence was established of a barrier for $\Omega_\omega$ of form $r^2 f(\theta), \lambda > 0$, on every cone $T_\Psi$, $\Psi < \pi$. The reader is recommended to take a look at that paper, at least at Sections 1 and 2. We limited our applications there mostly to the Dirichlet problem, preferring to defer the further applications, Phragmén-Lindelöf theorems, etc., to the present paper where the extremal supersolutions reveal the precise growth conditions.

Section 1 introduces notation and recalls some facts about the maximizing operator. In Section 2 we prove existence and continuous dependence on $\lambda$ and $\omega$ of a unique (extremal supersolution) $F_{\omega, \lambda}$ (i.e. $M_\omega(r^2 F_{\omega, \lambda}(\theta)) = 0$) for all $\omega$ and $\lambda$, $0 < \omega \leq 1$, $-\infty < \lambda < \infty$. The proof depends upon an existence, uniqueness, and stability theorem of the author's [9] for an ordinary differential equation with a certain nonlinear but monotone singularity at the origin. We then restrict $\lambda > 0$ and $n \geq 3$ and analyze the first zero, $\Psi(\lambda, \omega)$ of $F_{\omega, \lambda}$, finding that it is a continuous strictly decreasing function of $\lambda$ with domain $(0, \infty)$ and range $(0, \pi)$.

In Section 3 we consider Phragmén-Lindelöf theorems on cones. Such theorems are really just extended maximum principles in which the boundedness of the domain $\Omega$ is removed but is replaced by a growth condition at $\infty$ on the function $u$. The growth condition $o(r^{\Psi(\lambda, \omega)})$ obtained for the cone $T_\Psi$, where $\lambda(\Psi, \omega)$ denotes the inverse function of the extremal zero $\Psi(\lambda, \omega)$, is best possible, as shown by the solution $r^2 F_{\omega, \lambda}(\theta), \lambda = \lambda(\Psi, \omega)$, which is zero on the boundary of $T_\Psi$. As a special case we obtain the $o(r^1)$ growth condition on the half space $T_{\pi/2}$, a classical result for Laplace's equation and a result obtained for uniformly elliptic equations by Gilbarg [2] when $n = 2$ and by Hörmander [6] when $n \geq 3$. We show, however, that only in the case $T_\Psi = T_{\pi/2}$ is the growth condition $o(r^{2(\Psi, \omega)})$ for $\Omega_\omega$ the same as the classical growth condition $o(r^{2(\Psi, \omega)})$ for Laplace's equation. In fact, we show by example in Theorem 6 that the classical growth condition on $T_\Psi, \Psi = \pi/2$, does not suffice even with the additional restriction that $L$ have continuous coefficients tending at $\infty$ to those of the Laplacian. However, Friedman [1] had shown in the case $n = 2$ that the classical growth condition $o(r^{2(\Psi, \omega)})$ does suffice provided that the coefficients of $L$ are Dini continuous at $\infty$; results were also obtained on cones for $n \geq 3$ but the growth condition used was not precise. Other results of Phragmén-Lindelöf type on a half space (where the $o(r^1)$ growth condition is simpler and independent of uniform ellipticity) can be found in [17], [3], and [4].

Section 4 establishes that the solution of a Dirichlet problem has a certain Hölder continuity at a boundary point provided the boundary values