On the successive minima of a bounded star domain.

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Dedicated to Max Dehn.

Sunto. - È dato nel seguente capoverso.

Let \( F(X) \) be a bounded distance function and \( \Lambda \) an arbitrary lattice in the plane. Let further \( P, Q \) run over all pairs of independent points of \( \Lambda \) for which

\[
F(P) \leq F(Q).
\]

We call

\[
\mu_1(\Lambda) = \min F(P), \quad \mu_2(\Lambda) = \min F(Q)
\]

the two successive minima of \( \Lambda \) and denote by \( M \) the upper bound of \( \mu_1(\Lambda)\mu_2(\Lambda) \) extended over all lattices of a fixed given determinant. I prove in this paper that there exists at least one lattice for which this upper bound is attained.

§ 1. Points and lattices.

Let \((x_1, x_2)\) be rectangular coordinates in the Euclidean plane. We identify the point \( X = (x_1, x_2) \) of these coordinates with the vector \( X \) of components \( x_1, x_2 \) and use the usual vector notation. Thus if

\[
X = (x_1, x_2) \quad \text{and} \quad Y = (y_1, y_2)
\]

are any two points, then

\[
|X| = \sqrt{x_1^2 + x_2^2}
\]

denotes the distance of the point \( X \) from the origin

\[
O = (0, 0)
\]

or the length of the vector \( X \). Further

\[
|X, Y| = x_1y_2 - x_2y_1
\]

is the determinant of \( X \) and \( Y \), and, for real \( u, v \), \( uX + vY \) is the point

\[
uX + vY = (ux_1 + vy_1, ux_2 + vy_2).
\]
Assume, in particular, that \( X \) and \( Y \) are independent, i.e. that
\[
|X, Y| = 0.
\]
Then the set \( \Lambda \) of all points
\[
P = uX + vY, \quad \text{where} \quad u, v = 0, \mp 1, \mp 2, \ldots,
\]
is a lattice, and the positive number
\[
d(\Lambda) = | |X, Y||
\]
is the determinant of this lattice; the points \( X, Y \) form a basis of, or generate, the lattice.

If \( t 
eq 0 \) is real, then \( t\Lambda \) denotes the lattice of all points \( tP \) where \( P \) runs over \( \Lambda \). Evidently \( t\Lambda \) and \( -t\Lambda \) are the same lattice, and
\[
d(t\Lambda) = t^2d(\Lambda).
\]

\section{Star domains.}

Let
\[
F(X) = F(x_1, x_2)
\]
be a (bounded, symmetrical) distance function, i.e. a function of \( X \) with the following properties:

(A) \( F(O) = 0 \): \( F(X) > 0 \) if \( X \neq 0 \).
(B) \( F(tX) = |t|F(X) \) for all real \( t \) and for all points \( X \).
(C) \( F(X) \) is a continuous function of \( X \) (i.e. of \( x_1, x_2 \)).

The inequality
\[
K: \quad F(X) \leq 1
\]
then defines a (bounded, symmetrical) star domain \( K \), i.e. a point set \( K \) in the plane with the following properties:

(A) \( K \) is bounded and closed, and contains \( O \) as an inner point.
(B) Every line through \( O \) meets \( K \) in a finite line segment of which \( O \) is the centre.
(C) The boundary \( C: F(X) = 1 \) of \( K \) is a Jordan curve.

The more general inequality,
\[
cK: \quad F(X) \leq c,
\]
where \( c > 0 \), defines a star domain \( cK \) similar to \( K \); it consists of the points \( cX \) where \( X \) runs over \( K \).

Since \( K \) is a bounded set, there exist \( K \)-admissible lattices \( \Lambda \), i.e. lattices which contain no inner points of \( K \) except \( O \). Denote by
\[
\Delta(K) = l. b. d(\Lambda)
\]