Oscillation and Nonoscillation of Delay Differential Equations (*).

HIROSHI ONOSE (Mito city, Japan)

Summary. — Oscillation and nonoscillation theorems are presented for the second order retarded differential equation

\[(r(t)x'(t))' + f(x(g(t)), t) = 0 \quad (r(t) > 0)\].

Our aim here is to get more natural shapes of the results of Kusano and Naito [Annali di Matematica pura ed applicata]. This enables us to know how work \(r(t)\) for the oscillatory properties of (A).

I. — Introduction.

Consider the second order delay differential equation

\[(A) \quad (r(t)x'(t))' + f(x(g(t)), t) = 0\]

and its special case

\[(B) \quad (r(t)x'(t))' + x(g(t))F\left([x(g(t))]^2, t\right) = 0\].

For these equations the following conditions are assumed to hold throughout the paper

(a) (i) \(r(t)\) is continuous and positive for \(t > 0\) and

\[g(t) = \int_{t}^{\infty} \frac{ds}{r(s)} < \infty \quad \text{for all } t > 0,\]

or

(ii) \(\lim_{t \to \infty} R(t) = \infty\), where \(R(t) = \int_{t}^{t} \frac{ds}{r(s)}\);

(b) \(f(y, t)\) is continuous for \(|y| < \infty, t > 0\) and \(zf(z, t) > 0 \quad (z \neq 0)\) for \(t > 0\);

(c) \(g(t)\) is continuous for \(t > 0\), \(g(t) \leq t\) and \(\lim_{t \to \infty} g(t) = \infty\).

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Many authors studied the oscillatory properties of equation (B) which is classified according to the nonlinearity of the continuous function $yF(y^2, t)$ with respect to $y$ (Cf. [3], [7], [8]). Here, we discuss the oscillatory properties of more general equation (A).

We restrict our attention to solutions of (A) which exist on some ray $[t_0, \infty)$ and are nontrivial in every neighborhood of infinity. A solution is said to be oscillatory if it has arbitrarily large zeros, otherwise, it is said to be nonoscillatory. Equation (A) itself is called oscillatory if all of its solutions are oscillatory.

For the results for (B) or (A), we refer the reader to [1], [2], [3], [4], [6], [9].

The purpose of our paper is to establish some oscillation and nonoscillation theorems for equation (A).

2. - Nonoscillation and oscillation theorems for the case $g(t) < \infty$.

We first mention the following

**Lemma 2.1** [3]. - If $x(t)$ is a positive solution of (A), then it satisfies

\begin{equation}
k \geq x(t) \geq -r(t)x'(t)g(t),
\end{equation}

where $k$ is a constant, for all sufficiently large $t$, and $x'(t) \geq 0$ or $x'(t) \leq 0$ eventually.

**Proof.** - By the assumptions, there is $t_1$ such that $x(g(t)) > 0$ for $t > t_1$. From (A), we get easily that

\begin{equation}
(r(t)x'(t))' \leq 0 \text{ for } t \geq t_1
\end{equation}

and that

\begin{equation}
x(t) - x(t_1) \leq r(t_1)x'(t_1) \int_{t_1}^{t} r^{-1}(s) \, ds.
\end{equation}

From (2) and (3), we have the lemma.

**Theorem 1.** - Suppose that

\begin{enumerate}
\item[(d)] $f(y, t)$ is nondecreasing in $y$ for all $y > 0$ and $t > 0$,
\item[(e)] $f(y, t)/y$ is nonincreasing for all $y > 0$ and $t > 0$.
\end{enumerate}

Then, a necessary and sufficient condition for (A) to have a nonoscillatory solution which is asymptotic to a nonzero constant as $t \to \infty$ is that

\begin{equation}
\int_{c}^{\infty} g(t)f(c, t) \, dt < \infty \text{ for some } c > 0.
\end{equation}