Parametrized $\partial$ Operator on Pseudoconvex Sets (*) (**).

Claudio Rea (L’Aquila)

§ 1. — Introduction.

For a given smoothly parametrized operator $P_t$ of constant degree there is generally no continuously parametrized solution of the equation $P_t u_t = f_t \in RP_t$, even when $f_t$ is regular with respect to the pair parameter-variable. A simple example is given by the equation $u_t + t^2 u_t = (t^2 x - y) u_x + (t^2 y + x) u_y = t$ on the plane. In the present paper we show that the above fact does not happen if $P_t$ is the $\partial$ operator on any strongly pseudoconvex set [prop. (4.1)] and we give a smoothly parametrized solution for the Stein case (*). We work on a $C^\infty$ family $\mathcal{W}$ of complex manifolds $X_t$; the space of the parameter $t$ is a ball $B$ of $\mathbb{R}^m$ and could be reduced to a segment without any real loss of generality. Consider the equation

$$\overline{\partial}v = f$$

(it should be written $\overline{\partial}(v_t = f_t)$) on forms with values in a $C^\infty$ vector bundle $E$ such that $E_t = E|X_t$ is a holomorphic bundle (precise details about the algorithm are given in next section). In section 6 we shall prove the following

(1.2) Theorem. — Let $\mathcal{W} = \{X_t\}_{t \in B}$ be a $C^\infty$ family of complex manifolds with $X_0$, Stein. Each compact set $K \subset X_0$ has a neighborhood $\Omega$ in $\mathcal{W}$ such that, for every smooth family $(f_t)_{t \in B}$ of $C^\infty (p, q + 1)$-forms with values in the fiber bundle $E_t$ which satisfies $\overline{\partial}f_t = 0$, there exists on $\Omega$ a $C^\infty$ family $(v_t)_{t \in B}$ of $(p, q)$-forms which resolves (1.1).

Equation (1.1) is considered here with vector bundle valued forms out of necessity and not for excessive, apparent generality. Actually our proof goes through the existence of a continuous solution for the holomorphic tangent bundle which is, by the way, just of the type of $E$.

(*) Entrata in Redazione il 20 maggio 1975.
(**) Supported by C.N.R. research groups.
(*) Indeed our manifold at time 0 need not to be Stein but only «limit of Stein» (i.e. every compact set has a Stein neighborhood), but since we do not know whether this hypothesis is really more general, we assume it to be Stein.
In section 7 we give two applications of this theorem:

- pseudorigidity of Stein manifolds is proved for \( C^\infty \) deformations,
- two proofs of the complex Frobenius theorem of L. Nirenberg are given.

The first one is based on pseudorigidity and Newlander-Nirenberg theorem; the second one (only sketched) is a direct application of the present methods to Kohn proof ([7]) of the integrability theorem for almost complex structures.

§ 2. - a) Our total space is the product of a connected differential manifold \( X \) and an open ball \( B \) of \( \mathbb{R}^n \). At each point \((x,t)\) of \( X \times B \) is defined a tensor \( J: T_xX \to T_xX \) with \( J^2 = -id \) which depends smoothly on \((x,t)\) and satisfies the integrability conditions which are necessary (and sufficient) to give each \( X \times \{ t \} \) the structure of a complex manifold. We will note this complex manifold by \( X_t \) and the product \( X \times B \) with this additional structure by \( \mathcal{W} \). \( \mathcal{W} \) is a \( C^\infty \) family of complex manifolds. It is useful to introduce the trivial map \( \pi: X \times B \to \mathcal{W} \) and the natural projection \( \pi: \mathcal{W} \to B \). By the Frobenius-Nirenberg theorem [8] the structure of family of complex manifolds given by the tensor \( J \) can be described by an atlas \((W_i, \chi_i)\) of diffeomorphisms

\[ \chi_i: W_i \to \text{open subset of } \mathbb{C}^n \times \mathbb{R}^n \]

in such a way that the \( t \)-component of \( \chi_i(x, t) \) is \( t \), the coordinates \( z_h \) are holomorphic functions of \( z_i \) in \( W_i \cap W_h \) for fixed \( t \) and the complex structure on \( X \times \{ t \} \) is just the same that was induced by \( J \), i.e. \( X_t \).

Let \( \varphi_0 \in C^\infty(X_0) \) be the function, plurisubharmonic at \( \{ \varphi_0 = 0 \} \), defining \( \Omega_0 = \{ \varphi_0 < 0 \} \) and \( \varphi \) be its trivial extension to \( \mathcal{W} \) given by \( \varphi(\xi(x, t)) = \varphi_0(x) \). The restriction \( \varphi_0 \) of \( \varphi \) to \( X \), defines a set \( \Omega_0 \subset X \), which is obviously still strongly pseudoconvex for \( |t| \) small enough, say \( |t| < \delta \). Thus we can suppose, up to an unessential restriction, that the basis ball \( B \) is just \( |t| < \delta \). The open subset \( \Omega \) of \( \mathcal{W} \) given by \( \varphi < 0 \) is the neighborhood of \( \Omega_0 \) mentioned in theorem (1.2).

If \( \Omega_0 \) is Stein, a classical procedure shows that, up to a convenient restriction of \( B \), by taking a sufficiently increasing and convex function \( F: \mathbb{R} \to \mathbb{R}, \varphi = F(\varphi) \) and the minimum eigenvalue \( \lambda_\psi \) of its Levi form can be assumed to be greater than any given continuous function on \( \Omega \), [9].

b) The complexified tangent bundle of \( \mathcal{W} \) has a sub-bundle \( T \) of vectors which are linear combinations of \( \partial \partial z_1, \ldots, \partial \partial z_n \). The restriction \( T_i \) of \( T \) to \( X_i \) is the usual complex tangent bundle. Let \( A^{1\bar{1}} \) be the dual bundle of \( T \) and \( A^{1\bar{1}} \) its conjugate bundle. At each point \( \zeta \in \mathcal{W} \), the fibers \( A^{1\bar{1}}(\zeta) \) and \( A^{1\bar{1}}(\zeta) \) generate an exterior algebra

\[ \bigoplus_{\nu=0}^n A^{\nu\bar{\nu}}(\zeta) \]

The vector space \( A^{\nu\bar{\nu}}(\zeta) \) is the fibre of a bundle \( A^{1\bar{1}} \) over \( \mathcal{W} \) whose restriction \( A^{1\bar{1}} \) to \( X_i \) is the usual bundle of \((p, q)\)-forms on \( X_i \). Note that \( A^{1\bar{1}} = A^{1\bar{1}} \otimes A^{1\bar{1}} \).

Let \( E \) be a complex \( C^\infty \) vector bundle on \( \mathcal{W} \) whose restriction \( E_i \) to \( X_i \) is a holo-