TWO-DIMENSIONAL $\gamma, \beta$ DISTRIBUTION

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An analytical and graphical investigation of the one-dimensional $\gamma, \beta$-distribution was described and performed in [1]. The present paper reports the results of an investigation of the two-dimensional $\gamma, \beta$-distribution. Such investigations are necessary because of potentially wide applications of the distribution for the stochastic description of signals in radiation measurement systems, accident situations in different fields of science and technology, two-dimensional signals in radioelectronics, metrology, and pattern recognition, in reliability and queueing theory, and so on. In all cases the two-dimensional distribution possesses the following general, mandatory properties: description of continuous random variables bounded on one side and random variables in a prescribed interval; different form for describing different random variables; distribution must be general for some particular cases, continuous, and infinitely divisible. The two-dimensional $\gamma, \beta$-distribution, first proposed and partially investigated in [2], satisfies these conditions and requirements. Some comparisons of estimates of such a distribution are presented in [3]. Despite the scientific importance and potential wide application, the $\gamma, \beta$-distribution has still not been adequately investigated analytically and has not been studied at all graphically.

Density of the Two-Dimensional $\gamma, \beta$-Distribution. When the random variable $Y$ is described by the $\gamma$-distribution and the random variable $X$ with $Y = \text{const}$ is described by the $\beta$-distribution, according to the data in [3] the density has the form

$$p(x, y) = \frac{a \cdot e^{-\frac{(y-x)^q}{a}}}{\text{Beta}(b, \gamma) \cdot \Gamma(b)} \Gamma(b, \gamma) \Gamma(b, \gamma) \Gamma(b, \gamma) \frac{y^{b-p-q} \cdot e^{-\frac{-(y/a)^p}{b}}}{\Gamma(p-b) \Gamma(q) \cdot \text{Beta}(b, \gamma)}$$

(1)

where $a, b, c, p, q$ are positive parameters of the distribution. The region of existence of the distribution density where $p(x, y) > 0$ is, in the general case, described by positive random variables $0 < k < X < Y < \infty$, where $k$ is the location parameter. The regions of existence of $p(x, y)$ for two values of $k$ ($k = 0, k > 0$) are shown in Fig. 1. This corresponds to the condition of the distribution itself. As $k$ increases, the density of the distribution shifts into the region of high values of $X$ and $Y$. The regions of existence presented above reveal some features of the distribution and make it possible to establish correctly the limits of integration in determining different parameters of the distribution.

The element of probability $p(x, y) dx dy$ is the probability of falling within an elementary rectangle with sides $dx$ and $dy$ next to the point $(X, Y)$. This probability equals the volume of the elementary parallelepiped bounded above by the surface $p(x, y)$ and resting on the elementary rectangle $dx dy$. As $x \to y, x \to 0, y \to x,$ and $y \to \infty$ the density of the distribution approaches zero.

Analysis of Figs. 2 and 3 shows that as the parameters $p, q, b, c$ increase, the distribution density at the maximum increases rapidly and the bell-shaped surface moves upwards and becomes narrower at the sides. As the parameters $p, q, b, c$ increase, the bell-shaped surface becomes wider along the abscissa and narrower along the ordinate.

One-Dimensional Distributions. To obtain the density of the distribution of $X$ it is necessary to integrate $p(x, y)$ according to Eq. (1) over the entire range of the random variable $x < y < \infty$. This is impossible to do in general form because the factor $\exp[-(y/a)^p]$ is nonintegrable. For this reason, we shall find the desired distribution density $p(x)$ for four values $c = 1, 2, 3$, and 4 with $c = 1$.

$$p_1(x) = \frac{a \cdot e^{-\frac{(y-x)^q}{a}}}{\text{Beta}(b, \gamma) \cdot \Gamma(b)} \Gamma(b, \gamma) \Gamma(b, \gamma) \Gamma(b, \gamma) \frac{y^{b-p-q} \cdot e^{-\frac{-(y/a)^p}{b}}}{\Gamma(p-b) \Gamma(q) \cdot \text{Beta}(b, \gamma)}$$

(2)

where $\text{Beta}(1-q, 1-b+p, -(x/a))$ and $\text{Beta}(1 + b - p - q, 1 + b - p, -(x/a))$ are confluent hypergeometric functions.
Similarly, we obtain for $c = 2$

\begin{align*}
 p_2(x) &= R \left[(y-x)^{q-1}y^{2b-\theta+\gamma} \exp \left[-(y/a)^2\right] \right] dy = R \left[2a^{-2b}x^{2b} \Gamma(p+q-2b) \right]^{-1} x^{2b-1} \times \\
 \times & \left\{ a^{-p} x^p \Gamma(p+q-2b) \Gamma\left(\frac{2b-p}{2}\right) \left(\frac{1+2b-p-q}{2}, \frac{1}{2}\right), \left(\frac{2+q-2b}{2}, \frac{1}{2}\right), -(x^2/a^2) \right\} + \\
 + & 2a^{-2b}x^{2b}\Gamma(p-2b)\Gamma(q) F_q \left[0, \frac{1+2b-p-q}{2}, \frac{1+2b-p}{2}, \frac{2+q-2b}{2}, -(x^2/a^2) \right] + \\
 + (1-q) \left\{ a^{-1-p} x^{1+p} \Gamma(p+q-2b) \Gamma\left((2b-p-1)/2\right) \times \\
 \times & F_q \left[0, \frac{2-q}{2}, \frac{3-q}{2}, \frac{3+q-2b}{2}, -(x^2/a^2) \right] \right\}.
\end{align*}

(3)

According to Eqs. (1)-(3), the distribution density can be expressed in terms of special functions, i.e., the $\gamma$ and $\beta$ function and the confluent and general hypergeometric functions.

Dividing the two-dimensional distribution density $p(x, y)$ in Eq. (1) by the density $p_1(x)$ in Eq. (2) with $c = 1$, we obtain the conditional distribution density of $Y$ under the condition that the random variable $X$ assumes a prescribed value $x$:

\begin{align*}
 p_1(y|x) &= x^{q-1}(y-x)^{q-1} \exp\left(-(y/a)^2\right) \left\{ (1/a)^{-b+p} \Gamma(b-p) \right\} F_i \left[1-q, 1-b+p, -(x/a) \right] + \\
 + (1/x)^{-b+p} \Gamma(-b+p) \Gamma(q) F_i \left[1+b-p-q, 1+b-p, -(x/a) \right] \right\} \left(\frac{b+p+q}{2}\right) \right\].
\end{align*}

(4)