RICHARD A. WEIDA

AN EXTENSION OF BRUEN CHAINS

ABSTRACT. One way to obtain a new non-Desarguesian translation plane is by constructing a new spread that is not subregular. Chains of reguli in a regular spread of PG(3, q) were first introduced by Bruen as a method of obtaining a non-subregular spread. In this paper, we shall extend Bruen's notion of a chain of reguli. Let $\Omega$ be a regular spread of PG(3, q). A collection of reguli in $\Omega$ such that every line of $\Omega$ is contained in exactly none or two of these reguli will be called a nest of reguli. Let $\Gamma$ be the spread obtained by replacing in $\Omega$ the lines of the nest with the lines of some other partial spread of PG(3, q) covering the same points. We shall show that in the case where the number of reguli in the nest is no more than $q$, $\Gamma$ is not subregular and its full collineation group is the inherited group.

1. INTRODUCTION

Bruck and Bose [12] (see also Andrè [7]) introduced the notion of spreads in finite projective spaces of odd dimension and gave a construction which generates all finite translation planes from the collection of all such spreads. Thus the problem of creating new translation planes has been reduced to the construction of new spreads. Spreads of PG(3, q) are those which have been most studied. The resulting translation planes have order $q^2$ and are called 'two-dimensional' over their kernels since their coordinatizing quasifields can be represented as two-dimensional vector spaces over their left-operator skew-fields.

One approach to the construction of a new spread in PG(3, q) is to start with a regular spread $\Omega$, find an 'interesting' partial spread $U$ in $\Omega$, and then (hopefully) replace the lines of $U$ with those of some other partial spread $V$ which cover exactly the same set of points. An example of such a partial spread is the set of lines of a chain of reguli, first studied by Bruen [14]. Bruen showed that, in this case, $V$ consists of $(q+1)/2$ lines of each of the opposite reguli and that the resulting spread is a non-Desarguesian translation plane whose full collineation group is the group inherited from $\Omega$. Bruen also constructed two such spreads: for $q = 5$ and 7. Since then other such spreads have also been constructed for $q = 7, 9, 11, 13$ and 17 (see [5], [6], [9], [15], [16], [17] and [18]). New spreads can be derived from many of these spreads by reversing disjoint reguli (see [4], [10] and [20]).

In this paper, we shall generalize the notion of chains by increasing the size (i.e. the number of reguli) of the configuration. In particular, when the number of reguli in our configuration is between $(q+3)/2$ and $q$, inclusive, we are able to show that whenever the associated partial spread $U$ is replaceable, the
replacement set \( V \) must consist of opposite half-reguli. The spread \( S \) obtained by replacing \( U \) with \( V \) is not subregular, at least for \( q \geq 7 \). Furthermore, the full collineation group of the associated translation plane will be its inherited group, for \( q > 7 \).

This new class of (hopefully) replaceable partial spreads is extremely rich. Various examples of nests which are not chains are given in [8], [9] and [20]. Furthermore, Baker and Ebert [8] have shown the existence of a reversible nest for all odd primes \( q \). It has been shown by Oakden [19] that there are exactly 13 projectively different regulus-containing spreads in \( \text{PG}(3, 5) \). All of these may be found by reversing various combinations of nests and reguli. In the same manner over 50 different regulus-containing spreads have been found in \( \text{PG}(3, 7) \) (see [20] for more complete details).

2. Preliminary results

Let \( \Sigma = \text{PG}(3, q) \) denote the projective three-space over the finite field \( \text{GF}(q) \). A spread \( S \) of \( \Sigma \) is a collection of \( q^2 + 1 \) mutually disjoint lines, which necessarily partition the points of \( \Sigma \). A regulus \( R \) of \( \Sigma \) is a set of \( q + 1 \) mutually skew lines such that any transversal to three lines of \( R \) is a transversal to every line of \( R \). The set of transversals to \( R \) form another regulus called the opposite regulus to \( R \), denoted \( R' \). Note that the lines of \( R \) and the lines of \( R' \) cover exactly the same set of points of \( \Sigma \). We further note that three skew lines of \( \Sigma \) determine a unique regulus, and a regulus is determined by any three of its lines.

We call a spread \( S \) regular if for any triple of lines in the spread, it contains the regulus determined by those lines. A well-known result states that a spread \( S \) is regular if and only if the resulting translation plane \( \pi(S) \) is Desarguesian [13]. Furthermore, if \( m \) is any line of \( \Sigma \) not in the regular spread \( S \), then the \( q + 1 \) lines of \( S \) which meet \( m \) form a regulus, denoted by \( R(m) \). Clearly, \( m \) is a line of the opposite regulus \( R(m') \).

**Definition.** A nest of reguli is a collection \( P \) of reguli in a regular spread \( \Omega \) of \( \Sigma = \text{PG}(3, q) \) such that every line of \( \Omega \) is contained in exactly zero or two reguli of \( P \). The size \( t \) of a nest \( P \) is the number of reguli in \( P \).

Clearly, a chain of reguli is a nest of size \((q + 3)/2\). In [8], Baker and Ebert have explicitly shown the existence of nests of size \( q \), for \( q \) any odd prime. Furthermore, they have shown that the partial spread \( U \) contained in \( \Omega \) that consists of the lines of the reguli of these nests is always replaceable. For particular values of \( q \), the author has constructed reversible nests of various other sizes (see [9] and [20]).