A NUMERICAL INVESTIGATION OF THE ONE-DIMENSIONAL NEWTONIAN THREE-BODY PROBLEM

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Abstract

Numerical orbit integrations have been conducted to characterize the types of trajectories in the one-dimensional Newtonian three-body problem with equal masses and negative energy. Essentially three different types of motions were found to exist. They may be classified according to the duration of the bound three-body state. There are zero-lifetime predictable trajectories, finite lifetime apparently chaotic orbits, and infinite lifetime quasi-periodic motions. The quasi-periodic orbits are confined to the neighbourhood of Schubart’s stable periodic orbit. For all other trajectories the final state is of the type binary + single particle in both directions of time. The boundaries of the different orbit-type regions seem to be sharp. We present statistical results for the binding energies and for the duration of the bound three-body state. Properties of individual orbits are also summarized in the form of various graphical maps in a two-dimensional grid of parameters defining the orbit.

1 Introduction

The great astrophysical significance of the general three-dimensional gravitational three-body problem has motivated extensive numerical simulations (e.g. Hut & Bahcall 1983, Anosova 1986), as well as theoretical studies (Heggie 1975, Hut 1983). Attempts to base the statistical theory of gravitational interactions on a more general principle have been made by Monaghan and Nash (Monaghan 1976a,b, Nash & Monaghan 1978). However, the general problem has so far proved to be too complicated for a comprehensive analytical and/or numerical analysis. For this reason it is necessary to study the simpler systems to gain intuition for the investigation of the full problem. The simplest Newtonian three-body problem is perhaps that in one dimension, which seems to be rather unexplored, although studies of a similar system with the infinite sheet potential have been published (Matulich & Miller 1986). Despite the ‘toy model’ nature of the one-dimensional system, a full understanding of it will give useful information and ideas for studies of the general three-body problem.

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In this paper we study the one-dimensional three-body problem with equal masses and negative energy. The selection of initial values can be reduced to the choice of two parameters and the entire orbit-space can be covered using a two-dimensional grid of parameters. We have calculated a set of orbits in a 100 x 180 grid. For each grid-point there is another grid-point corresponding to the same orbit but with reversed direction of initial velocity, and we obtain the entire orbit for all times by combining these two cases. Thus we have data for a totality of 9000 orbits over the time interval from $-\infty$ to $+\infty$.

2 Initial conditions and numerical method

2.1 Regularization of the Hamiltonian

If we fix the masses and the gravitational constant to be equal to one, then the Hamiltonian of the problem is given by

$$H = \frac{1}{2} \sum_{i=0}^{2} \frac{1}{|x_i - x_j|} + \sum_{i<j} \frac{1}{|x_i - x_j|},$$

where $w_i$ are the momenta canonical to $x_i$. We number the bodies so that $x_1 \leq x_0 \leq x_2$, and because the bodies cannot change their order we can introduce the (positive) distances $q_i$ as new coordinates:

$$q_1 = x_0 - x_1, \quad q_2 = x_2 - x_0.$$

We are not interested in the centre of mass motion, thus we may use the generating function $S = p_1(x_0 - x_1) + p_2(x_2 - x_0)$ to obtain the new Hamiltonian

$$H = p_1^2 + p_2^2 - p_1 p_2 - \frac{1}{q_1} - \frac{1}{q_2} - \frac{1}{q_1 + q_2}.$$  

(3)

To regularize the equations of motion we adopt the Aarseth-Zare (1973) method, which is ideal here because the middle body can be used as the reference body with respect to which the other two are regularized. In the present case the necessary additional transformation is $q_1 = Q_1^2$, $q_2 = Q_2^2$, i.e. the generating function is $S = p_1 Q_1^2 + p_2 Q_2^2$, which gives the new momenta $P_i = 2 Q_i p_i$. Together with the time transformation $t' = q_1 q_2$ this gives the regularized Hamiltonian $\Gamma = t'(H - E)$ in the form

$$\Gamma = \frac{1}{4}(P_1^2 Q_2^2 + P_2^2 Q_1^2 - P_1 P_2 Q_1 Q_2) - Q_1^2 - Q_2^2 - Q_1^2 Q_2^2/(Q_1^2 + Q_2^2) - Q_1^2 Q_2^2 E.$$

(4)

Here $E$ is the numerical value of the Hamiltonian $H$, calculated from initial values.

2.2 Selection of initial values

To characterize the orbit one has to specify four initial values. We restrict ourselves to the case of negative total energy, and thus we may take $E = -1$. For every orbit the ratio of the distances $q_1/q_2$ takes any pre-assigned value at some moment of time (in future or past). We start the integration when this ratio is 1, i.e. $q_1(0) = q_2(0) = R$. [The same orbit