SPECTRUM OF OPERATORS IN IDEAL SPACES

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One considers "weighted translation" operators in ideal Banach spaces. It is proved that if the translation is aperiodic (the set of periodic points has measure zero), then the spectrum of such an operator is rotation-invariant. This result can be extended (under certain additional restrictions) to "weighted translation" operators acting in regular subspaces of ideal spaces, in particular, to operators in Hardy spaces.

In this note we prove the rotation-invariance of the spectrum of aperiodic operators of "weighted translation" in ideal* spaces and uniform B-algebras.†

THEOREM 1. Let \((X, \mu)\) be a space with a positive \(\sigma\)-finite measure, let \(E\) be the ideal space of measurable functions on \(X\), and let \(\varphi\) be a measurable mapping of \(X\) onto \(X\) such that:

1) \(\mu(\varepsilon) = 0 \iff \mu(\varphi(\varepsilon)) = 0\);

2) the translation operator \(T_{\varphi}\) is defined and bounded in \(E\). Assume that the set of \(\varphi\)-periodic points in \(X\) has measure zero and that \(M_{L^\infty}(X, \mu)\). Then the spectrum of the operator \(T = MT_{\varphi}\) is rotation-invariant.

Proof. Assume that \(\lambda\) belongs to the continuous or to the point spectrum of \(\sigma(T)\), and let \(|\lambda| = 1\). Let us show that \(\lambda \in \sigma(T)\). For any measurable set \(F\), \(F \subseteq X\), we define the sets \(Q_i(F), i = 0, \ldots\), by the equalities \(Q_0(F) = F\), \(Q_i(F) = \varphi(Q_{i-1}(F))\). We fix a natural number \(N\). By virtue of the aperiodicity of \(\varphi\) there exist (see [4]) sets \(B_1, \ldots, B_N\) with the properties:

1) the sets \(Q_i(B_i), i = 1, \ldots, N, k = 0, \ldots, N+1\), are pairwise disjoint;

*Regarding ideal spaces and the corresponding terminology, see [1, p. 91].

†For operators induced by ergodic transformations in \(L^1\), the corresponding result has been obtained by Petersen [2], while for operators with unitary spectrum in uniform \(B\)-algebras — by the author [3].
2) \( q_{N+i}(B_i) = \bigcup_{i=0}^{N+i} \tilde{B}_i \);
3) \( \bigcup_{i=0}^{N+i} \tilde{B}_i \), is a set of complete measure in \( X \).

We define the function \( q_N, q_n \in L^\infty(X, \mu) \), by the equalities

\[
q_N(x) = \alpha_i, \quad x \in q_i(B_i),
\]

where \( \alpha_i \) is the \((N + i - 1)\)-th root of unity closest to \( \alpha \). Obviously \( \|T_q q_n - \alpha q_n \|_\infty \leq \frac{2\pi}{N} \).

We consider the function \( f \in E \), such that \( \|f\| = 1 \), \( \|T_q f\| < 1/N \). Then

\[
\|T_q q_n - \alpha f q_n\| = \|(T_q f) - \alpha f q_n\| < \left( 2\pi + 1 \right) \cdot N^{-i}.
\]

Since \( \|q_n\| = 1 \) and \( N \) is arbitrarily large, we have \( \alpha \in \sigma(T) \).

Remark. In Theorem 1, instead of a \( \sigma \)-finite measure \( \mu \), it is sufficient to assume that the carrier of any function from \( E \) is a set of \( \sigma \)-finite measure.

**THEOREM 2.** Assume that \( X, \mu, E, \varphi \) satisfy the conditions of Theorem 1, let \( L \) be a subspace of \( E \) consisting of functions with an absolutely continuous norm,* let \( A \) be a closed subalgebra in \( L(X, \mu) \), and let \( M \subseteq A \). In addition, assume that:

1) the subspace \( L \) is invariant relative to the operator \( T_q \); 
2) the algebra \( A \) is invariant relative to \( T_q \); 
3) for any measurable set \( F, F \subseteq X \), and for any \( \varepsilon, \varepsilon > 0 \) there exist \( \tilde{f}, \tilde{f} \in A \) and \( \tilde{\varphi}, \tilde{\varphi} \subseteq F \) such that \( \mu(F \setminus \tilde{\varphi}) < \varepsilon, \|\tilde{f} - \tilde{\varphi}\| = 1 \) a.e. on \( \tilde{\varphi} \) a.e. on \( |\tilde{f}| < 1 \setminus X \setminus \tilde{\varphi} \).

Then the spectrum of the restriction \( T/L \) is rotation-invariant.

For the proof of Theorem 2 one needs the following

**LEMMA.** Assume that \( \varphi \) is nonsingular, i.e., \( \mu(\varphi) = 0 \Rightarrow \mu(\varphi(\epsilon)) = 0 \). Then there exists a set \( B \) and sets \( B_{j, N}, i = 1, 2, \ldots, \kappa(N), N = 1, 2, \ldots \), with the properties: 1) the sets \( q_i(B), i = 0, 1, \ldots \), and the sets \( q_i(B_{j, N}), i = 0, 1, \ldots, \kappa(N) \), are pairwise disjoint and their union is a set of complete measure in \( X \); 2) \( \kappa(j, N) \) is \( N^2 \), \( j = 1, 2, \ldots, \kappa(N) \), and \( \varphi_{\kappa(j, N) + 1}(B_{j, N}) < \bigcup B_{j, N} \); 3) \( \bigcup B_{j, N + 1} < \bigcup B_{j, N} \), and \( \mu(\bigcup B_{j, N}) \rightarrow 0 \) \((N \rightarrow \infty)\).

The proof of this lemma is based on considerations similar to those used in [4] and it will not be given here.

**Proof of Theorem 2.** Let \( f \in L \), \( \|f\| = 1 \), \( \|T_q f\| < \varepsilon \) and \( |d| = 1 \). Let \( p \) be a natural number such that

\[
(1 - \varepsilon)^p < \varepsilon. \quad (1)
\]

Making use of the lemma and of the fact that \( f \) is a function with an absolutely continuous norm, one can choose \( N \), \( N > 2p \), such that \( \|T_{\tilde{f}} f\| > \frac{1}{N} \), where \( \tilde{f} \) is the union of the sets \( q_i(B_{j, N}), j = 0, 1, \ldots, \kappa(N), i = p, p + 1, \ldots, \kappa(N) + 1 \). From now on the index \( N \) will be omitted in the notations \( B_{j, N}, k(j, N) \). By the conditions of the theorem one can select sets \( \tilde{B}_j \subseteq B_j \) and functions \( q_j \in A \) such that

\[
|q_j| = 1 \quad \text{on} \quad \tilde{B}_j, \quad (2)
\]

\[
\|T_{q_j} f - (T_{q_j} q_j) \|_{L^\infty} < \frac{\varepsilon}{N}, \quad \kappa = \sum_{j=0}^{\kappa(N)} k(j), \quad j = 0, \ldots, \kappa(N), \quad i = 0, \ldots, k(j), \quad (3)
\]

*That is, \( \|q_n f\| \nrightarrow 0 \) if \( F_{n+1} \subseteq F_n \) and \( \bigcap_{n=1}^{\infty} F_n = \emptyset \).