Nonelliptic Schrödinger Equations

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Summary. Nonelliptic Schrödinger equations are defined as multidimensional nonlinear dispersive wave equations whose linear part in the space variables is not an elliptic equation. These equations arise in a natural fashion in several contexts in physics and fluid mechanics. The aim of this paper is twofold. First, a brief survey is made of the main nonelliptic Schrödinger equations known by the authors, with emphasis on water waves. Second, a theory is developed for the Cauchy problem for selected examples. The method is based on linear estimates which are strongly related to the dispersion relation of the problem.

Key words. nonlinear Schrödinger equations, dispersive equations, blow-up, Cauchy problem

1. Introduction

This work is concerned with the Cauchy problem for equations of the type

$$i \frac{\partial u}{\partial t} + Lu + F(u)u = 0, \quad (1.1)$$

where $u = u(x, t)$ is a complex-valued function defined for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $L$ is a linear differential operator which is defined in Fourier variables (a hat indicates the Fourier transform) by

$$\hat{L}u(\xi) = p(\xi)\hat{u}(\xi). \quad (1.2)$$

Throughout this paper, the symbol $p(\xi)$ of $L$ will be assumed to be real. In many applications, $L$ is not elliptic (unlike the classical Schrödinger equation where $L$ is the Laplacian $\Delta$). Moreover, $L$ is not necessarily a second-order operator.
The mapping $F$ is (possibly nonlocal) real (though not always) and depends on $u$ and possibly on its spatial derivatives. For instance, in the context of water waves (e.g., in the Davey-Stewartson systems), the nonlocal part of $F$ arises via the equation for the induced mean flow velocity potential.

This paper is in some sense a continuation of a previous one [13] in which we studied the Cauchy problem for the Davey-Stewartson systems.

Our aim is to survey known or unknown properties for equations such as (1.1) and which do not depend on the ellipticity of $L$, or on the integrability of the equations. Actually, the models we consider here are generally not integrable. (See Fokas and Sung [44] for a nice survey of the rigorous existence results obtained via the I.S.T. for DS I, DS II and other integrable equations.) To avoid artificial generality we mainly restrict ourselves to systems stemming from physical situations. Consequently, Section 2 is devoted to a brief review of examples of nonelliptic Schrödinger equations arising in physical contexts where the dispersion plays a predominant role.

Section 3 is of a more mathematical nature. Here we study the Cauchy problem (including the possible blow-up of solutions in finite time) for some selected examples. They include generalizations of the Davey-Stewartson systems, the Benney-Newell systems for the evolution of weakly nonlinear modes in a dispersive medium, and a model of Shrira for the evolution of a three-dimensional packet of weakly nonlinear gravity waves. Our methods are based on linear estimates which are strongly related to the dispersion relation of the problem. They are recalled in an Appendix.

Notation. The norm in Lebesgue spaces $L^p(\mathbb{R}^n)$ will be denoted $\| \cdot \|_p$, while the norm in Sobolev spaces $H^s(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n), (1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^n) \}$ will be indicated as $\| \cdot \|_{s,2}$.

$L^p(0, T; X)$ denotes the space of (classes of) measurable $X$-valued functions $f$ such that $\| f \|_{L^p(0, T; X)} = \left( \int_0^T \| f(t) \|^p_X \, dt \right)^{1/p} < \infty$.

Finally, $C([0, T]; X)$ denotes the space of continuous functions $f : [0, T] \rightarrow X$ equipped with its natural norm, and $C_w([0, T]; X)$ means that $X$ is endowed with the weak topology.

2. Examples of Nonelliptic Schrödinger Equations

Benney and Newell [2] derived the system for the evolution of wave trains whose amplitudes vary slowly in both space and time. It reads

$$
\mu \left( \frac{\partial a_\xi}{\partial T} + \sum_r \frac{\partial \omega_\xi}{\partial k_\xi} \frac{\partial a_\xi}{\partial X_r} \right) = \frac{i \mu^2}{2} \sum_{r,s} \frac{\partial^2 \omega_\xi}{\partial k_\xi \partial k_\xi} \frac{\partial^2 a_\xi}{\partial X_r \partial X_s} + i \varepsilon \sum_{m,n} \alpha_{bmn} \overline{a_m} a_n + i \varepsilon^2 \sum_p B_{\xi} a_i |a_p|^2 + i \varepsilon^2 \sum_{q,r,s} \gamma_{qr} a_q a_r a_s, \quad (2.1)
$$

$\xi = 1, \ldots, N$. 