A generalization of the variation diminishing property

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For totally positive matrices, a new variation diminishing property on the sign of consecutive minors is obtained. This property is used to show shape preserving properties of curves generated by totally positive bases and, in particular, of B-spline curves.

1. Introduction

In curve design, preservation of shape properties is a common requirement. Usually a curve is designed using a "control polygon" which must resemble and even exaggerate the shape of the curve. A control polygon is given by the ordered set of control points, which are the coefficients of the parametrical curve with respect to a set of basic functions. It is well-known (cf. [4,2]) that control polygons with respect to totally positive bases enjoy shape preserving properties.

A totally positive basis is a basis whose collocation matrices are all totally positive. Let us recall that a matrix is totally positive (TP) if all its minors are non-negative. One of the most important consequences of total positivity which is closely related with those geometric properties is the variation diminishing property: If $T$ is a nonsingular TP matrix and $x$ a vector, then the number of changes of sign in the consecutive components of $Tx$ is bounded by the number of changes of sign in the consecutive components of $x$ (cf. chapter V of [6], section 5 of [1]). This can be expressed as $S^-(Tx) \leq S^-(x)$, where, for any vector $v = (v_1, \ldots, v_n)$, $S^-(v)$ denotes the number of sign changes of the sequence $v_1, \ldots, v_n$, ignoring zeros.

In this paper (in section 3) we shall derive several generalizations of the ordinary variation diminishing property in the sense that, given an $m \times n$ TP matrix $T$ and an $n \times r$ matrix $A$ satisfying some additional conditions, then the number of changes of sign in the consecutive $r \times r$ minors of $TA$ is bounded by the number of changes of sign in the consecutive $r \times r$ minors of $A$.

Local shape control is also important for designing a curve. This property explains why B-spline curves are preferred by many designers instead of Bézier,
although the representation in both cases is managed by a totally positive basis. Local shape control leads to matrices $T$ with a special structure of zeros, which will be called (strongly) $p$-restricted. In section 2 we obtain some ancillary results on factoring TP matrices of this kind.

On the other hand, some particular shapes of control polygons allow us to derive further shape preserving properties of the designed curve. In the case of having local shape control, the kind of control polygons which ensure the generalized variation diminishing properties (i.e., for minors) can be considerably enlarged. This idea leads to the definition of regular matrices of order $p$, which are considered in section 3.

In section 4, we give some applications of the results of section 3 concerning this generalized variation diminishing property to curves generated by totally positive bases and in particular to B-spline curves. In examples 4.5 and 4.6, we show that the number of local extrema in a linear combination of basic functions is bounded by the number of local extrema in the ordered set of coefficients. In example 4.8 we give an analogous result for the number of inflections of plane curves. For the special case of B-splines this was obtained in [5]. In example 4.9 we prove that the number of changes of sign of the torsion of the curve is bounded by the number of changes of sign of the torsion of the control polygon, provided that each $p$ consecutive points of the control polygon can be projected onto some plane to form a convex polygon.

2. Totally positive matrices which are $p$-restricted

It is well-known that a matrix $T$ is totally positive (TP) if and only if it can be written as a product of nonnegative bidiagonal matrices. In this auxiliary section we obtain more precise results on this factorization under certain conditions on $T$.

First, let us introduce some notations similar to those of [1]. Given $k, n \in \mathbb{N}$, $1 \leq k \leq n$, $Q_{k,n}$ will denote the set of all increasing sequences of $k$ natural numbers less than or equal to $n$. Let $A$ be a real square matrix of order $n$. For $k \leq n$, $l \leq n$, and for any $\alpha \in Q_{k,n}$ and $\beta \in Q_{l,n}$ we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of $A$ containing rows numbered by $\alpha$ and columns numbered by $\beta$. The conversion of the matrix $A = (A_{ij})_{1 \leq i,j \leq n}$ is the matrix $A^\#$ of order $n$ whose $(i,j)$ entry is $A_{n-i+1,n-j+1}$. It is straightforward that $A$ is TP if and only if $A^\#$ is TP and that $(AB)^\# = A^\# B^\#$.

**Definition 2.1**

We say that a matrix $T = (T_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is strongly $p$-restricted if any $p$ consecutive rows are linearly independent and vanish outside some $p$ consecutive columns, i.e., for any $k$, $1 \leq k \leq m - p + 1$, there is some $l$, $1 \leq l \leq n - p + 1$, such that for $k \leq i \leq k + p - 1$, $T_{ij} = 0$ for $j < l$ and $j > l + p - 1$ and det $T[k,k+1,\ldots,k+p-1|l,l+1,\ldots,l+p-1] \neq 0$. 