THE STRESS DISTRIBUTION IN AN ANISOTROPIC HALF-PLANE WITH TWO ELLIPTIC HOLES OR CRACKS

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By use of the method of complex potentials, conformal mappings and least squares this problem is reduced to solving a system of linear algebraic equations with respect to the unknown constants that occur in the required functions. We describe the results of numerical studies of the variation of the stress intensity factors for cracks in an anisotropic half-plane under tension of the half-plane and force on its boundary.

Two figures, two tables. Bibliography: 7 titles.

S. A. Kaloerov [1, 2] has used integrals of Cauchy type and expansions in Faber polynomials to solve a number of problems for a half-plane with two elliptic holes along or perpendicular to a rectilinear boundary. In the present paper we carry out similar studies using the method of linear coupling and a new approach [3] that makes it possible to study a more complicated geometry of the holes and the combination of them.

Consider an anisotropic lower half-plane $S$ with a rectilinear boundary $L_0$ and elliptic holes with semiaxes $a_l, b_l$ and boundary curves $L_t$ ($l = 1, 2$) (Fig. 1). At infinity the half-plane is subject to tensile forces $\sigma_x^\infty = p$. The boundaries of the holes are free of external loads, and there is a uniform pressure of intensity $q$ on the interval $c_l d_l$ of the rectilinear boundary.

Determining the stressed state of this half-plane reduces to finding the complex potentials [3]

$$\Phi_j'(z_j) = \Gamma_j - \frac{\mu_{j+1} q}{2\pi i} \ln \frac{z_j - d_j}{z_j - c_j} + \sum_{l=1}^{2} \sum_{n=1}^{\infty} \left\{ \varphi_{j,ln}(z_j) a_{j,ln} - \bar{\varphi}_{j,ln}(z_j) \bar{a}_{j,ln} - \bar{\varphi}_{j+1,ln}(z_j) \bar{a}_{j+1,ln} \right\},$$

(1)

where

$$\bar{\varphi}_{j,ln}(z_j) = -n / R_{jl} \left( \zeta_{jl}^2 - m_{jl} \right) \zeta_{jl}^{-1};$$

$$\bar{\varphi}_{j+1,ln}(z_j) = -n / R_{jl} \left( \zeta_{jl}^2 - m_{jl} \right) \zeta_{jl}^{-1};$$

$$\bar{\varphi}_{j,ln}(z_j) = \frac{\mu_j (1 - \mu_j) \mu_{j+1} \mu_{j+1} \mu_{j+1} a_{16} / 2a_{22}}{(\mu_j - \mu_j) (\mu_j - \mu_j) (\mu_j - \mu_j)} p,$$

when $\sigma_x^\infty = p$ at infinity; $\mu_j$ are the roots of the known characteristic equation [4]; $a_{ij}$ are the strain coefficients of the material of the half-plane; $\zeta_{jl}$ are variables defined by the implicit relations

$$z_j = z_{0jl} + R_{jl} (\zeta_{jl} + m_{jl} / \zeta_{jl}); \quad z_{0jl} = x_{0l} + \mu_j y_{0l};$$

$$R_{jl} = \left[ a_l (\cos \phi_l + \mu_j \sin \phi_l) + ib_l (\sin \phi_l - \mu_j \cos \phi_l) \right] / 2;$$

$$m_{jl} = \left[ a_l (\cos \phi_l + \mu_j \sin \phi_l) - ib_l (\sin \phi_l - \mu_j \cos \phi_l) \right] / 2 R_{jl};$$

$x_{0l}, y_{0l}$ are the coordinates of the centers of the ellipses $L_l$ in the $Oxy$ system; $\phi_l$ is the angle between the positive $Ox$-axis and the semiaxis $a_l$ of the ellipse $L_l$.

By satisfying the boundary conditions

$$2 \text{Re} \sum_{j=1}^{2} (-\mu_j, 1) \delta_j \Phi_j'(z_j) = (0, 0) \quad \text{on} \quad L_l \quad (l = 1, 2),$$

(2)
using the method of least squares [5], we obtain a system of linear algebraic equations to determine \( a_{j,m} \). After this system is solved, the complex potentials (1) become known, which makes it possible to compute the stresses in the half-plane [4]

\[
(\sigma_x, \sigma_y, \tau_{xy}) = 2\Re\sum_{j=1}^{2} (\mu_j^2, 1, -\mu_j) \Phi_j'(z_j),
\]

and, in the case of a crack (when one of the semiaxes of the ellipse equals zero), the stress intensity factor (SIF) as well [5]

\[
k_1 = \lim_{r \to 0} \sqrt{2r} \left( \sigma_x \sin^2 \varphi_1 + \sigma_y \cos^2 \varphi_1 - 2\tau_{xy} \cos \varphi_1 \sin \varphi_1 \right),
\]

\[
k_2 = \lim_{r \to 0} \sqrt{2r} \left[ (\sigma_y - \sigma_x) \cos \varphi_1 \sin \varphi_1 + 2\tau_{xy} \left( \cos^2 \varphi_1 - \sin^2 \varphi_1 \right) \right].
\]

Detailed numerical studies of the stress distribution and the variation of the stress intensity factors as functions of the geometric and elastic properties of the half-plane were performed. The results are in good agreement with known results computed by other methods [1, 2]. Some of the results are described below for the case of a half-plane made of a composite with an epoxy glue, reinforced by unidirectional graphite fibers [6], for which \( E_x = 14.9 \times 10^4 \text{ MPa}, E_y = 0.6 \times 10^4 \text{ MPa}, G_{xy} = 0.4 \times 10^4 \text{ MPa}, \nu_{xy} = 0.31 \) (material M1) or \( E_x = 0.6 \times 10^4 \text{ MPa}, E_y = 14.9 \times 10^4 \text{ MPa}, G_{xy} = 0.4 \times 10^4 \text{ MPa}, \nu_{xy} = 0.31 \) (material M2), and also for an isotropic half-plane (material M3). The results are shown up to factors of \( p \) and \( q \) in the case of force at infinity and uniform pressure on the interval \( c_1d_1 \) respectively. The values of \( k_1^+ \) and \( k_1^- \) refer respectively to the upper and lower ends of the crack.

Table 1 gives the values of \( k_1^\pm \) for a half-plane with one vertical crack of unit semi-length and a circular hole of unit radius (Fig. 2a), when the length of the bridge between the boundary of the half-plane and the crack \( c \) takes on various values. The distance between the edge of the crack and the hole was assumed to be 0.5. Studies were also carried out for a half-plane with two vertical cracks (Fig. 2b). As the computations showed, the values of the stress intensity factor for the upper crack agree almost perfectly with the data of Table 1.