CONVERGENCE OF TWO FUNCTIONALS OF ASYMPTOTICALLY NORMAL SUMS OF INDEPENDENT RANDOM VARIABLES

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We consider two functionals of sums of independent random variables and demonstrate that the validity of the central limit theorem for the sums of independent random variables that enter the arguments of those functionals is a sufficient condition for one of the functionals and a necessary and sufficient condition for the other one to have a weak limit.

V. M. Kruglov in [1], considering five functionals of sums of independent random variables, whose asymptotic analysis inspired the invention of the invariance principle, proved the equivalence of weak convergence of each of those functionals to the validity of the central limit theorem for the sums of independent random variables entering the functionals. Using the scheme of the proof proposed in [1], we can extend some results of A. M. Mark [2] to one of those five functionals.

1. Let a triangular array \( X_{n,k}, 1 \leq k \leq m_n, n = 1, 2, \ldots \), of independent inside each row random variables be given, which are asymptotically uniformly negligible:

\[
\max_{1 \leq k \leq m_n} \mathbb{P}(|X_{n,k}| > \varepsilon) \longrightarrow 0, \quad \varepsilon > 0. \tag{1}
\]

We denote

\[
S_{0}^{(n)} = 0, \quad S_{k}^{(n)} = \sum_{j=1}^{k} X_{n,j}, \quad S_{m_n}^{(n)} = \sum_{k=1}^{m_n} X_{n,k}, \quad 1 \leq k \leq m_n. \tag{2}
\]

Let \( p(t), t \in [0, 1], \) be a continuous function with at most numerable zeros. We fix a positive number \( \tau \) and introduce

\[
\alpha_{n,k} = \alpha_{n,k}(\tau) = \int_{|x|<\tau} x \, d\mathbb{P}(X_{n,k} < x),
\]

\[
A_{k}^{(n)} = \sum_{j=1}^{k} \alpha_{n,j}, \quad \sigma_{n,k}^{2} = \mathbb{E} \frac{(X_{n,k} - \alpha_{n,k})^{2}}{1 + (X_{n,k} - \alpha_{n,k})^{2}}, \quad \sigma_{n}^{2} = \sum_{k=1}^{m_n} \sigma_{n,k}^{2}.
\]

Without loss of generality, we assume that

\[
\sigma_{n,k}^{2} > 0, \quad 1 \leq k \leq m_n, \quad n = 1, 2, \ldots . \tag{3}
\]

If \( \sigma_{n,r}^{2} = 0 \) for some \( r, 1 \leq r \leq m_n \), then the summand \( X_{n,r} - \alpha_{n,r} = 0 \), i.e., \( X_{n,r} \) is degenerate. For each \( n = 1, 2, \ldots \), we partition the interval \([0, 1]\) by the points

\[
i_{0}^{(n)} = 0, \quad i_{k}^{(n)} = \sigma_{n}^{-2} \sum_{j=1}^{k} \sigma_{n,j}, \quad 1 \leq \lambda \leq m_n.
\]
We construct the random polygonal line $X_n(t), t \in [0, 1]$, with break points $(t_k^{(n)}, \gamma_k^{(n)} - A_k^{(n)})$. On the interval $[t_{k-1}^{(n)}, t_k^{(n)}]$ it is of the form

$$X_n(t) = S_{k-1}^{(n)} - A_{k-1}^{(n)} + \frac{t - t_{k-1}^{(n)}}{t_k^{(n)} - t_{k-1}^{(n)}}(X_{n,k} - a_{n,k}).$$

(4)

**Theorem 1.** Let $\tau > 0$ be given. Assume that the summands of (2) satisfy (1) and (3). For

$$\mathbb{P}\left\{ \sum_{k=1}^{m_n} p\left( \frac{\sigma_n^{-2} \sum_{j=1}^{k} \sigma_{n,j}^2}{\sigma_{n,k}^2} (S_k^{(n)} - A_k^{(n)})^2 < x \right) \right\} \Rightarrow V_2(x)$$

(5)

it is necessary and sufficient that

$$\mathbb{P}(S_{m_n}^{(n)} - A_{m_n}^{(n)}(\tau) < x) \Rightarrow \Phi(x),$$

(6)

where $\Phi(x)$ is the standard normal distribution function, $V_2(x)$ is the distribution function corresponding to the characteristic function

$$D(\lambda) = \int e^{i\lambda x} dV_2(x) = (D(2i\xi))^{-1/2},$$

$D(\lambda)$ is the Fredholm determinant corresponding to the integral equation

$$\lambda \int_0^1 \min(s, t)p(t)f(t) dt = f(s).$$

The proof of this theorem is based on the following lemma.

**Lemma 1.** The equality

$$V_2(x) = \mathbb{P}\left\{ \int_0^1 p(t)W^2(t) dt < x \right\}$$

(7)

is true, where $W(t), t \in [0, 1]$, is the Wiener process.

**Proof.** Let $\{X_n\}_{n \geq 1}$ be a sequence of independent identically distributed random variables with zero means and variances equal to one, $S_0 = 0, S_k = \sum_{j=1}^k X_j$. We denote by $X_n(t)$ the random polygonal line constructed as follows:

$$X_n(t) = \frac{1}{\sqrt{n}} \left( S_{k-1} + \frac{t - t_{k-1}}{t_k - t_{k-1}} X_{n,k} \right),$$

$$t_k = k/n, \quad t_{k-1} = \tau, \quad t_0 = 0, \quad t_n = 1.$$

By virtue of Donsker's theorem [5], the sequence of random polygonal lines $X_n(t), n = 1, 2, \ldots$, weakly converges to the Wiener process $W(t), t \in [0, 1]$. We introduce the random variable

$$Y_n = \int_0^1 p(t)X_n^2(t) dt.$$

By the just-mentioned Donsker's theorem,

$$\mathbb{P}(Y_n < x) \Rightarrow \mathbb{P}\left\{ \int_0^1 p(t)W^2(t) dt < x \right\}.$$

(8)

We represent the random variable $Y_n$ as

$$Y_n = \int_0^1 p(t)X_n^2(t) dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} p(t)X_n^2(t) dt = \frac{1}{n} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} p(t)\left( S_{k-1}^{(n)} - \frac{t - t_{k-1}}{t_k - t_{k-1}} X_{n,k} \right) dt.$$