ASYMPTOTIC ANALYSIS OF THE QUEUEING SYSTEM $G | G | 1 | \infty$
WITH GROUP SERVICING

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The objective of the research presented is to formulate and prove an assertion concerning the convergence of the waiting time in the $G | G | 1 | \infty$ system with group servicing to zero as $n \to \infty$.

1. Introduction

Queueing systems with group servicing are widely used in modeling computer networks, data transmission networks, wholesale systems, etc. (see [1]). Introducing group servicing into a single-channel system, we should consider arbitrary laws of distribution of service times and intervals between customer arrivals. We thus must investigate the $G | G | 1 | \infty$ systems whose analytic properties are rather complicated [2]. Then, it only remains for us to apply the statistical modeling [3]. But before we deal with this, it is appropriate to play with asymptotical methods [4]. The systems with group servicing allow us to choose a nonstandard asymptotics, namely, the mode where the group size $n$ tends to infinity.

2. Formulation of the Theorem

We assume that the recurrent input flow of the $G | G | 1 | \infty$ system is characterized by random intervals $b_i$ between the arrivals of the $i$th and $(i + 1)$th customers, $i \geq 1$. The random service time of the $i$th customer is $a_i$, $i \geq 1$. We assume that the random vectors $(a_i, b_i)$, $i \geq 1$, are independent identically distributed, consist of independent components, and satisfy the inequality $E c_i < 0$, $c_i = a_i - b_i$. (1)

Now we turn to a description of the service of groups of customers of size $n$, denoting by $A_j^{(n)} = \sum_{i=n(j-1)+1}^{nj} a_i$, $B_j^{(n)} = \sum_{i=nj}^{n(j+1)-1} b_i$, the service time of the $j$th group, and the interval between the times of formation of the $j$th and $(j + 1)$th groups, respectively. It is clear that the sequence $C_j^{(n)} = A_j^{(n)} - B_j^{(n)}$, $j \geq 1$, consisting of independent identically distributed random variables; moreover, in view of (1), $E C_j^{(n)} = n E c_i < 0$. (2)

Let $W_j^{(n)}$ stand for the time between the completion of formation of the $j$th group and the beginning of its servicing; then the recurrence relation $W_j^{(n)} = \max(0, W_j^{(n)} + C_j^{(n)})$, $j \geq 1$, is true. Setting $W_1^{(n)} = 0$, we get the Markov chain $W_j^{(n)}$, $j \geq 1$, which, in view of (1) and (2), possesses the limit distribution

$$\lim_{j \to \infty} P\{W_j^{(n)} > t\} = P\{W_j^{(n)} > t\}, \quad t \geq 0, \quad W^{(n)} = \sup\left\{0, \sum_{j=1}^{k} C_j^{(n)}, k \geq 1\right\};$$

(4)

moreover, the sequence $P\{W_j^{(n)} > t\}$, $j \geq 1$, monotonically increases as $j$ grows.

**Theorem 1.** We assume that for some $\mu > 0$ the inequality $E \exp\{\mu a_i\} < \infty$ (5)
is true (the Cramér condition). Then there exist positive $c, d, d < 1$, such that

$$P\{W^{(n)} > 0\} \leq cd^n, \quad n \geq 1.$$  \hspace{1cm} (6)

If, instead of the Cramér condition, for some $m > 2$ the inequality

$$Ea_i^m < \infty$$  \hspace{1cm} (7)

is true, then there exists a positive $q_m$ such that

$$P\{W^{(n)} > 0\} \leq \frac{q_m}{n^{m-1}}, \quad n \geq 1.$$  \hspace{1cm} (8)

3. Proof of the Theorem

We begin with the Cramér condition. In view of (1) and (5), we are able to choose a positive $\nu$ such that

$$E\exp\{\nu c_i\} = \nu < 1.$$  \hspace{1cm} (9)

We introduce the notation

$$S^{(n)}_k = \sum_{j=1}^{k} c_j^{(n)},$$

then formulas (4) and (9) yield

$$P\{W^{(n)} > 0\} = P\{\sup\{0, S_k^{(n)}, k \geq 1\} > 0\} \leq \sum_{k=1}^{\infty} P\{S_k^{(n)} > 0\}$$

$$\leq \sum_{k=1}^{\infty} E\exp\{\nu S_k^{(n)}\} \leq \sum_{k=1}^{\infty} f^k n = \frac{f^n}{1 - f}, \quad n \geq 1.$$  \hspace{1cm} (10)

Taking $c = 1/(1 - f), d = f$, we arrive at (6).

Now, let us turn to the consideration of the exponential case. We introduce $c'_i = a_i - b_{n+i}$; then

$$S_k^{(n)} = \sum_{i=1}^{n_k} c'_i = \sum_{i=1}^{n_k} \Delta c_i - nk_c,$$  \hspace{1cm} (11)

where $c = -Ec_i > 0$, $\Delta c_i = c'_i + c$. Taking condition (7) into account, we obtain

$$E(\max(0, \Delta c_i))^m < \infty.$$  \hspace{1cm} (12)

By analogy with (10),

$$P\{W^{(n)} > 0\} \leq \sum_{k=1}^{\infty} P\{S_k^{(n)} > 0\} = \sum_{k=1}^{\infty} P\left\{\sum_{i=1}^{n_k} \Delta c_i > nk_c\right\}.$$  \hspace{1cm} (13)

Using condition (12) and the results of [5], we are able to choose $n_m, g_m$ such that

$$P\left\{\sum_{i=1}^{n_k} \Delta c_i > nk_c\right\} \leq \frac{2nky_m}{(nk_c)^{m-1}} = \frac{2g_m}{r^{m-1}(nk)^{m-1}}$$  \hspace{1cm} (14)

for $n \geq n_m, k \geq 1$. Combining formulas (13) and (14), we arrive at

$$P\{W^{(n)} > 0\} \leq \frac{q_m}{r^{m-1}n^{m-1}}.$$  \hspace{1cm} (15)

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