AVERAGING AND HOPF BIFURCATION IN DEGENERATE SYSTEMS OF DIFFERENTIAL EQUATIONS WITH DELAYS

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This article considers bifurcation of families of periodic solutions from equilibrium states for systems of differential equations with constant delays that, upon linearization at zero, become systems of ordinary differential equations. An averaging transformation is constructed to simplify solution of the bifurcation problem.

The method of transforming equations to a form that is normal in some sense [1-3] has been successfully applied to solution of bifurcation problems on generation of cycles from equilibrium states of evolutionary equations. A feature of the bifurcation problem considered below is that it cannot be reduced to a two-dimensional bifurcation problem. We will show that the averaging method [4] can also be used here.

Consider the problem of bifurcation of a cycle of the system

\[ \frac{d}{dt}z(t) = A(z(0), z(t), z(t - \Delta)), \quad z(0) = 0, \quad \Delta > 0, \]  

from the equilibrium state \( s = 0 \). We assume that \( g : \mathbb{R}^n \to \mathbb{R}^n \) is a mapping that is smooth of class \( C^{r+4} \) \( r \geq 0 \), \( ||g(z_1, z_2, \alpha)|| = O(||z_1||, ||z_2||)^r \), and the spectrum \( A(\alpha) \) is subject to the usual conditions: It contains a pair \( \lambda(\alpha), \bar{\lambda}(\alpha) \) of complex conjugate eigenvalues of the form \( \lambda(\alpha) = \gamma(\alpha) + i\omega(\alpha) \), where \( \gamma(0) = 0, \gamma'(0) = \nu \neq 0, \omega(0) = \omega > 0 \), and the remainder of the spectrum is uniformly distant from the imaginary axis and located in the left half-plane.

We also assume that the relationship between \( A(\alpha) \) and \( \alpha \) is smooth of class \( C^{r+4} \).

In the first approximation, the system (1) of differential equations with a delay degenerates into a system of ordinary differential equations.

We represent \( z \in \mathbb{R}^n \) as \( z = (x, y) \). P \( \oplus Q = \mathbb{R}^2 \oplus \mathbb{R}^{n-2} \), in accordance with the spectrum \( A(0) \), so that \( P \) is the proper subspace corresponding to \( 0 \). \( \lambda(0) \), and \( Q \) is its complement. This factorization allows us to write

\[ A(\alpha) = \begin{pmatrix} A_P(\alpha) & 0 \\ 0 & A_Q(\alpha) \end{pmatrix}, \quad \text{where } A_P(\alpha) = \begin{pmatrix} \gamma(\alpha) & -\omega(\alpha) \\ \omega(\alpha) & \gamma(\alpha) \end{pmatrix}, \]  

and the spectrum \( A_Q(\alpha) \) is uniformly distant from the imaginary axis.

We select \( \varepsilon > 0 \) and rescale the equation by means of the substitutions \( x \to \varepsilon x, \alpha \to \varepsilon \alpha, \) and \( y \to \varepsilon y \). Using Taylor's formula (in the new coordinates), we obtain the system

\[
\frac{d}{dt}x(t) = A_P(\alpha)x(t) + \varepsilon G(x(t), y(t)), \quad y(t) = O(\varepsilon^2),
\]

where \( G : \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}^2 \) is a homogeneous polynomial of degree 2.

We now represent \( x \) in polar coordinates, \( x = (r \cos \theta, r \sin \theta) \). When we make the transition to the new time \( t = (1 + q)r \), \( t < 1 \) system (2) takes the form

\[
(1 + q)^{-1} \frac{d}{dr}r(\tau) = \varepsilon r(\tau) + \varepsilon C_2 + \varepsilon^2 C_3 + \varepsilon G_2 + O(\varepsilon^2) + O(\varepsilon); \]

1There is an error in the title of Belan's article in Dinamicheskoe Sistemy, Vol. 11, 1992. It should read "On quasiperiodic solutions of semilinear parabolic equations."

\[(1 + q)^{-1} \frac{d}{d\tau} \rho(\tau) = \omega + \varepsilon D_1 + O(\varepsilon^2) + O(z^2);
\]
\[(1 + q)^{-1} \frac{d}{d\tau} y(t) = A_Q y(\tau) + \varepsilon F_2 + O(\varepsilon^2 n) + O(z^2).\]  \hspace{1cm} (3)

where

\[
(C_k, D_1, F_2) = (C_k, D_1, F_2)(r(\tau), r \left( r - \frac{\Delta}{1 + q} \right), \rho(\tau), \frac{\Delta}{1 + q});
\]

\[
G_2 = G_2 \left( \frac{\Delta}{1 + q}, \frac{\Delta}{1 + q}, y(r), y(\tau), \frac{\Delta}{1 + q}, \frac{\Delta}{1 + q} \right).
\]

The functions on the right side of system (3) are related to the right sides of (2) by the relations

\[
C_k(r, \rho, \phi, \psi) = \cos \phi B_2^k(r \cos \phi, r \sin \phi, \rho \cos \psi, \rho \sin \psi) + \sin \phi B_2^k
\]

\[
\times (r \cos \phi, r \sin \phi, \rho \cos \psi, \rho \sin \psi);
\]

\[
G_2(r, \rho, \phi, \psi, y_1, y_2) = \cos \phi G^1(r \cos \phi, r \sin \phi, \rho \cos \psi, \rho \sin \psi, y_1, y_2)
\]

\[
+ \sin \phi G^2(r \cos \phi, r \sin \phi, \rho \cos \psi, \rho \sin \psi, y_1, y_2);
\]

\[
D_1(r, \rho, \phi, \psi) = \frac{1}{r} (\cos \phi B_2^1(r \cos \phi, r \sin \phi, \rho \cos \psi, \rho \sin \psi) - \sin \phi B_2^1(r \cos \phi, r \sin \phi, \rho \cos \psi, \rho \sin \psi));
\]

\[
F_2(r, \rho, \phi, \psi) = J_2(r \cos \phi, r \sin \phi, \rho \cos \psi, \rho \sin \psi).
\]  \hspace{1cm} (4)

We should note that the right side of system (3), which corresponds to the variable \(\phi\), is not defined at the point 

\((0, \rho, \phi, \psi)\), just as the functions \(r, \rho, \phi, \psi\) are generally not even continuously continuable at this point. However, when \(\rho = r\), this function, like the function \((r, \phi, \psi)\) admits a continuation that is smooth of class \(C^{r+3}\) at the point 

\((0, \phi, \psi)\). This will be important below. We now turn to transforming the system of differential equations with delay 

(3). In system (3) we perform the substitution

\[
r = \rho + \varepsilon \rho^2 u_1(\theta) + \varepsilon^2 \rho^3 u_3(\theta);
\]

\[
\rho = \theta + \varepsilon \rho u(\theta);
\]

\[
y = h + \varepsilon \rho^2 w(\theta).
\]  \hspace{1cm} (5)

whose coefficients we will determine below. We write the transformed system in the form

\[
(1 + q)^{-1} \frac{d}{d\tau} \rho(\tau) = \varepsilon \left[ \alpha \rho(r) + C_2 - \rho^2 \frac{d u_1(\theta)}{d\theta} \right] + O(\varepsilon^2 n) + O(z^2);
\]

\[
+ \varepsilon' \left[ G_2 + \varepsilon C_3 - \rho^2 \frac{d u_1}{d\theta} D_1 + 2\varepsilon \rho u_1(\theta) C_2 + \rho^2 \frac{d u_2}{d\theta} \right] + O(\varepsilon^3) + O(z^2);
\]

\[
(1 + q)^{-1} \frac{d}{d\tau} \theta(\tau) = \omega + \varepsilon \left[ D_1 - \rho^2 \frac{d v}{d\theta} \right] + O(e^\alpha) + O(\varepsilon^2);
\]

\[
(1 + q)^{-1} \frac{d}{d\tau} h(\tau) = A_Q h(\tau) + \varepsilon \left[ F_2 + \rho^2 A_Q w - \rho^2 \frac{d w}{d\theta} \right] + O(e^\alpha) + O(\varepsilon^2).\]  \hspace{1cm} (6)

We now turn to determination of the coefficients of transformation (5). For a given \(F(\rho, \phi, \psi, \omega, \Delta)\) we write \(F_2(r, \rho, \phi, \psi, \omega, \Delta)\). By (4), it follows from the representation of \(F_2\) and the properties of \(J_2\) that we can write \(F_2(\rho, \phi, \omega, \Delta) = r^2 f_2(\phi)\). For \(u(\phi)\) we choose a periodic solution of the equation \(\frac{dw}{d\theta} = A_Q w + f_2(\phi),\) which exists by virtue of the spectrum \(A_Q\). For the function \(D_1(\rho, \phi, \psi)\) we can write \(D_1(\rho, \phi, \psi) = r d_1(\phi)\). \(d_1 = \frac{1}{2} \int_0^{2\pi} d_1(\theta)d\theta = 0\). For \(e(\phi)\) we choose a periodic solution of the equation \(\frac{de}{d\theta} = d_1(\phi)\). It follows from the representation (4) for \(C_2\) that we may have \(C_2(r, \phi) = C_2(\phi) \cdot r^2.\) \(\hat{e}_2 = 0\). For the function \(u_1(\phi)\) we take a periodic solution of the equation

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