FLUCTUATIONS IN ONE-DIMENSIONAL DYNAMIC SYSTEMS

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We consider one-dimensional dynamic systems with random fluctuations that are encountered in applications, analyze solutions, and investigate the stability of stationary points.

In applications we frequently encounter dynamic systems with one-dimensional phase spaces. Examples are provided by coastal systems in geomorphology, autonomous populations in biology, autocatalytic reactions in chemistry, etc. [1, 2]. In general, such systems are described by equations of the form

\[
\frac{dw}{dt} = F(w),
\]

where

\[ w \in [a, b]; \quad a, b \in \mathbb{R}. \]  

In such systems, an important role is often played by a variety of fluctuations that are as likely to be associated with fluctuations in macroscopic parameters as the stochastic nature of processes that occur in the systems themselves. Following [2], we will attempt to account for random fluctuations by replacing the deterministic equation (1) with the stochastic differential equation (a stochastic Ito equation)

\[
dw = F(w)dt + \sigma dw(t),
\]

where the second term describes the contribution of fluctuations to \( w \), and \( \sigma \) is the amplitude of the fluctuations (which we assume to be constant).

On \( dw(t) \) we impose the conditions

\[
\left\{ \frac{d}{dt}(w(t)) \right\} = 0, \\
\left\{ \left( \frac{d}{dt}(w(t)) \right)^2 \right\} = \sigma^2.
\]

where \( \left\{ \ldots \right\} = \int_0^\infty \rho(w, t | w_0, t_0) dw \) is the statistical average, and \( \rho(w, t | w_0, t_0) \) is the probability that the system coordinate in the phase space at time \( t \) will have the value \( w \) if it has the value \( w_0 \) at time \( t_0 \).

Averaging (2), we obtain Ito's equation for the mean value [2]:

\[
\frac{dw}{dt} = \langle F(w) \rangle.
\]

From Eq. (2) and the statistical independence of \( dw(t) \) and \( w \) we obtain, for the probability distribution function, the Itô-Fokker-Planck equation [2]:

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial w}(F(w)\rho) + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial w^2}.
\]

The formal solution of the Cauchy problem for this equation is of the form [3]

\[
\rho(w, t) = \exp \left\{ (t - t_0) \left[ \frac{\sigma^2}{2} \frac{d^2}{dw^2} - \frac{d}{dw}F(w) \right] \right\} \varphi(w),
\]

where \( \varphi(w) \equiv \rho(w, t_0) \) is the initial condition.

We now introduce the notation \( p \equiv -i \frac{d}{dw} \), so the function \( \rho(w, t) \) takes the form of the amplitude of the probability (\( \Psi \)-function) for a one-dimensional quantum particle with coordinate \( w \) and Hamiltonian \( H = \frac{p^2}{2\gamma} + pF(w) \) [4]. Thus, every one-dimensional stochastic system can be treated as a one-dimensional quantum particle with imaginary mass \( i\gamma^{-2} \) that is subject to friction. It now follows, according to [4], that \( \rho(w, t) = |w|^2 \), where \( |w|^2 \) is the eigenvector of the coordinate operator of a quantum particle, and \( |t\rangle \) is the state vector of the particle, which depends on time and is equal to

\[
|t\rangle = \exp \{-iH(t - t_0)\}|t_0\rangle \equiv \exp \{-i(H_0 + H_1)(t - t_0)\}|t_0\rangle.
\]
where \(|t_0\rangle\) is the state vector of the particle at time \(t_0\).

\[
H_0 = \frac{g^2}{2\hbar^2} I^2, \quad H_1 = pF(w). \quad \langle w|t_0\rangle \equiv \phi(w).
\]

We represent \(|t\rangle\) in the form

\[
|t\rangle = \exp\{-iH_0(t-t_0)|t_0\rangle + U(t)|t_0\rangle;
\]

now, upon differentiation of expression (8), we obtain, for the operator \(U(t)\), the equation

\[
\frac{dU}{dt} = -i(H_0 + H_1)U - iH_1 \exp \{-iH_0(t-t_0)\},
\]

from which it follows that

\[
U(t) = \exp \{-iH_0(t-t_0)\} \int_{t_0}^{t} \exp \{-iH_0(\theta - t_0)\} 
\times [-iH_1U(\theta) - iH_1 \exp \{-iH_0(\theta - t_0)\}] \, d\theta;
\]

this last is clearly equivalent to the integral equation

\[
\left\{1 + i \int_{t_0}^{t} d\theta \exp \{iH_0(\theta - t)\} H_1(\theta) \ldots \right\} U(t) = -i \int_{t_0}^{t} d\theta \exp \{iH_0(\theta - t)\}
\times H_1 \exp \{-iH_0(\theta - t_0)\}
\]

whose solution is

\[
U(t) = \sum_{n=0}^{\infty} (i)^{n+1} \int_{t_0}^{t} d\theta_n \int_{t_0}^{\theta_n} d\theta_{n-1} \ldots \int_{t_0}^{\theta_2} d\theta_1 \int_{t_0}^{\theta_1} d\theta_0 \exp \{iH_0(\theta_n - t)\} H_1(\theta_n)
\times \exp \{iH_0(\theta_{n-1} - \theta_n)\} H_1(\theta_{n-1}) \ldots \exp \{iH_0(\theta_0 - \theta_1)\} H_1(\theta_0) \exp \{-iH_0(\theta_0 - t_0)\}.
\]

Here we have used the known expansion for the operators \((I - A)^{-1} = \sum_{n=0}^{\infty} A^n\), where \(A\) is an operator and \(I\) is the identity operator. As a result,

\[
\rho(w, t) = \langle w|t \rangle = \langle w|\exp \{-iH_0(t-t_0)\}|t_0\rangle + \langle w|U(t)|t_0\rangle
\]

\[
= \exp \left\{(t-t_0) \frac{g^2}{2\hbar^2} \right\} \phi(w) + \sum_{n=0}^{\infty} (i)^{n+1} \int_{t_0}^{t} d\theta_n \int_{t_0}^{\theta_n} d\theta_{n-1} \ldots \int_{t_0}^{\theta_2} d\theta_1 \int_{t_0}^{\theta_1} d\theta_0 \exp \{iH_0(\theta_n - t)\} H_1(\theta_n)
\times \exp \{iH_0(\theta_{n-1} - \theta_n)\} H_1(\theta_{n-1}) \ldots \exp \{iH_0(\theta_0 - \theta_1)\} H_1(\theta_0) \exp \{-iH_0(\theta_0 - t_0)\}
\times \phi(y) + \sum_{n=0}^{\infty} \int_{t_0}^{t} d\theta_n \int_{t_0}^{\theta_n} d\theta_{n-1} \ldots \int_{t_0}^{\theta_2} d\theta_1 \int_{t_0}^{\theta_1} d\theta_0 \int_{t_0}^{\theta_n} d\theta_n \ldots \int_{t_0}^{\theta_1} d\theta_0 \exp \{iH_0(\theta_n - t)\} H_1(\theta_n)
\times \exp \{iH_0(\theta_{n-1} - \theta_n)\} H_1(\theta_{n-1}) \ldots \exp \{iH_0(\theta_0 - \theta_1)\} H_1(\theta_0) \exp \{-iH_0(\theta_0 - t_0)\}
\times \frac{\exp \left\{\frac{(w-y)^2}{2g^2(t-t_0)}\right\}}{g^2 \sqrt{2\Pi(t-t_0)^3/2}} [w - J_n] F(J_n - \frac{J_n}{g^2(t-t_0)^{3/2}})
\times \frac{\exp \left\{\frac{(J_n - J_{n-1})^2}{2g^2(t-t_0)}\right\}}{g^2 \sqrt{2\Pi(t-t_0)^3/2}} [J_n - J_{n-1}] F(J_{n-1}) 
\times \frac{\exp \left\{\frac{1}{2g^2(t-t_0)^{3/2}}\right\}}{g^3 \sqrt{2\Pi(t-t_0)^3/2}} [J_n - J_{n-1}] F(J_{n-1}) 
\times \frac{\exp \left\{\frac{1}{2g^2(t-t_0)^{3/2}}\right\}}{g^3 \sqrt{2\Pi(t-t_0)^3/2}} [J_n - J_{n-1}] F(J_{n-1}).
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