ESTIMATE FOR THE SECOND-ORDER DERIVATIVES OF SOLUTIONS OF CURVATURE-TYPE EQUATIONS

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A local estimate for the second-order solutions to the curvature-type equations is established.

Bibliography: 9 titles.

§1. Introduction

Let $S^n$ be the space of real symmetric $n \times n$-matrices, and let a function $F \in C^2$ be defined in a domain $D_0 \subset S^n$. We introduce the sets

\[ D_1 = \{ r \in S^n : F(r + \eta) \geq F(r) \quad \forall \eta \in S^n, \eta \geq 0 \}, \]

\[ D_2 = \{ r \in S^n : F(r) \text{ is concave} \}. \]

We denote by $D$ the convex connected component of the intersection $D_1 \cap D_2$. We assume that $D$ is nonempty and $I \in D$, where $I$ is the unit matrix. In addition, we assume that if $r \in D$, then $r + \eta \in D$ for any nonnegative matrix $\eta$.

For $F$ we can take the following functions (cf. [1, 2]):

\[ F(r) = S_{m}^{1/m}(r), \quad F(r) = S_{m,e}(r) \equiv \left( \frac{S_m}{S_e}(r) \right)^{1/(m-1)}, \]

where $S_m = \text{spur}_m r$. For these functions we have

\[ D = K_m = \{ r \in S^n : S_i(r) \geq 0, i = 1, \ldots, m \}. \]

In $D$, we assume that the function $F$ satisfies the following condition which is stronger than (1):

\[ \lambda \sum_{k=1}^{n} \frac{\partial F}{\partial r_{kk}} |\xi|^2 \leq \sum_{k=1}^{n} \frac{\partial F}{\partial r_{ij}} \xi_i \xi_j \leq \Lambda \sum_{k=1}^{n} \frac{\partial F}{\partial r_{kk}} |\xi|^2 \]

for all $r \in D$, $\xi \in \mathbb{R}^n$, where $\lambda, \Lambda, \mu$, and $\nu_0$ are constants. Hereinafter, we assume that all constants are positive.

We consider a positive-definite matrix function $A(x, z, p) \in S^n$, $(x, z, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. For functions $u \in C^2(\Omega)$ we introduce the notation

\[ u_i = \partial u / \partial x^i, \quad u_z = (u_i), \quad u_{xx} = (u_{ij}), \quad u_{xx} = A^{-1/2}(x, u, u_z) \cdot u_{xx} \cdot A^{-1/2}(x, u, u_z). \]

A function $u \in C^2(\Omega)$ is said to be admissible if $u_{xx} \in D$ at every point $x \in \Omega$.

In this paper, we study solutions to the equation

\[ F(u_{xx}) = f(x, u, u_z), \quad x \in \Omega, \]


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where $f \in C^2(\Gamma), \Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n$. In [3], for solutions of Eq. (4) the following estimate of the gradient of a matrix $A = (A^{ik})$ of the form

$$A^{ik}(p) = (a^2 + p^2)^{1-s/2} \left( s^k - (s-2) \frac{p^j p^k}{a^2 + (s-1)p^2} \right), \quad s > 1, \ a > 0,$$

is established in $\Omega$. In the sequel, we assume that the matrix $A$ and the function $f$ are defined in $B_R = \{(x, z, p) \in \Gamma : [z] + |p| < R\}$ and that the matrix $A$ is positive-definite, i.e., $\nu|\xi|^2 \leq A^{ij} \xi^i \xi^j \leq \nu^{-1}|\xi|^2, \xi \in \mathbb{R}^n$, where $R$ and $\nu$ are positive constants.

§2. Main Results

Starting our investigation, we planned to use the methods developed in [4] in order to obtain a local estimate of $|u_{xx}|$ for admissible solutions to Eq. (4). Trudinger [4] deals with fully uniformly elliptic equations of the general form $F(x, u, u_x, u_{xx}) = 0$ under the assumption that $F$ is concave with respect to the last variable. As was shown in [4], under the conditions

$$|F(x, u, p, 0)| \leq \mu_1 \lambda, \quad |F_x|, |F_u|, |F_p| \leq \mu_1 \lambda(1 + |r|);$$

$$|F_{xx}|, |F_{uu}|, |F_{xu}|, |F_{ux}|, |F_{pp}|, |F_{up}|, |F_{pu}| \leq \mu_1 \lambda(1 + |r|),$$

where $\mu_1 = \mu_1(|u| + |p|, \lambda/\lambda)$, the solution $u$ satisfies the inequality

$$|u|_{2,\Omega} \leq C(\lambda, \Lambda, \mu_1, \|u\|_{C^1(\Omega)}),$$

where $|u|_{2,\Omega} = \max_{\Omega}(\text{dist}^2 \{x, \partial\Omega\} : |u_{xx}(x)|)$.

Analyzing the methods presented in [4], we find that such methods can be applied to our equations without any restriction of the form (i), (ii); moreover, they lead to the estimate (5) neither in our case nor for the equations discussed in [4], i.e., the estimate (5) has not been proved. In Sec. 3, we reproduce the corresponding arguments from [4] and arrive at the following assertion.

**Theorem A.** Let $u \in C^4(\Omega)$ be an admissible solution to Eq. (4). For any $\nu > 0$, there exists a constant

$$M_2 = M_2(d, \nu, \lambda, \mu_1, \|f\|_{C^2(\Gamma)}, \|u\|_{C^1(\Omega)})$$

such that the inequality $\max_{\Omega} |u_{xx}| > M_2$ implies that the function $|u_{xx}(x)|$ takes its largest value at a point $x \in \Omega - \Omega(d)$, where $\Omega(d) = \{x \in \Omega : \text{dist}\{x, \partial\Omega\} > d\}$.

We give another formulation.

**Theorem B.** Let $u \in C^4(\Omega)$ be an admissible solution to Eq. (4), and let $|u_{xx}|$ take its largest value at a point $x_0$; moreover, $\text{dist}\{x_0, \partial\Omega\} = d > 0$. Then

$$\max_{\Omega} |u_{xx}| \leq M_2.$$

The arguments of [4] lead to similar assertions but not to the inequality (5). In order to use Theorems A and B effectively, we need to complete them by an estimate in $\Omega \setminus \Omega(d)$ for some $d > 0$. We can obtain this result by combining the central idea of [4] with the Krylov–Safonov method of obtaining estimates for the Hölder constant of the second-order derivatives (cf. [5]).