A RAPID METHOD OF SOLVING DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

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We propose and justify a new method of solving differential equations with small parameter, a method that fulfills the requirements of the problems of celestial mechanics. In contrast to the classical power-series method, the proposed method converges rapidly. We discuss new formulations of problems and the promise of the method.

Introduction. The classical power-series method is widely used at present in science and engineering to solve systems of nonlinear differential equations with a small parameter. We recall the outline of the method using the example of the simplest nonlinear differential equation. Consider the Cauchy problem

\[
\frac{dy}{dx} = y + \varepsilon y^2, \quad y(0) = 1.
\]

We seek a solution of the problem (1), (2) in the form of a series

\[
y = \sum_{k=0}^{\infty} \varphi_k(x) \varepsilon^k
\]

with undetermined coefficients \( \varphi_k(x) \). We substitute this series into Eq. (1), equate the coefficients of like powers of the parameter \( \varepsilon \), and obtain equations for sequentially determining the coefficients \( \varphi_k(x) \):

\[
\frac{d\varphi_0}{dx} = \varphi_0, \quad \varphi_0(0) = 1, \\
\frac{d\varphi_1}{dx} = \varphi_1 + \varphi_1^2, \quad \varphi_1(0) = 0, \\
\frac{d\varphi_2}{dx} = \varphi_2 + 2\varphi_0\varphi_1, \quad \varphi_2(0) = 0, \\
\ldots
\]

By sequentially solving these linear equations, we determine the coefficients \( \varphi_k(x) \) of the series uniquely. A similar scheme is applied for general systems of differential equations. The only difference is that the computations of the unknown column coefficients are more complicated. In the nineteenth century this asymptotic-series method was applied systematically to solve problems in celestial mechanics. The great French mathematician Henri Poincaré [4] applied it to determine the orbits of the planets of the solar system. He studied the question of the Cauchy problem for a system of ordinary differential equations

\[
\frac{dY}{dx} = F(x, Y, \varepsilon), \quad Y(0) = Y_0.
\]

where the system (3) has a special form (a Hamiltonian system of equations), and he solved the problem (3), (4) by the asymptotic-series method. It turned out, however, that this method not only converges badly: for some problems it may even diverge. The classical method that had been fruitfully applied in many natural sciences proved to be ineffective in celestial mechanics. The need arose to construct a rapid method that would be convergent in order to meet the requirements of the new problems. The object of the present article is to construct such a method, justify it, and give a practical implementation of it.

The basic idea of the new method is quite simple. We shall explain it using the example of one differential equation
\[ \frac{dy}{dx} = f(x, y, \varepsilon), \]  
with initial conditions
\[ y(0) = y_0. \]  
where \( 0 \leq \varepsilon \leq \varepsilon_0 \) is a small parameter. Suppose the problem (5), (6) has been solved uniquely in a certain interval with \( \varepsilon = 0 \). Assume that the solution \( y = \varphi(x) \) of the problem is known for \( \varepsilon = 0 \). We now take the argument \( x \) as parameter and the parameter \( \varepsilon \) as the new argument. Then the problem (5), (6) can be reduced to integrating the differential equation
\[ \frac{\partial y}{\partial \varepsilon} = \int_0^\lambda \left( \frac{\partial f(\xi, y, \varepsilon)}{\partial y} \frac{\partial y}{\partial \varepsilon} + \frac{\partial f(\xi, y, \varepsilon)}{\partial \varepsilon} \right) d\xi. \]  
Equation (7) can be solved by the following scheme:
\[ \frac{\partial y_n}{\partial \varepsilon} = \int_0^\lambda \left( \frac{\partial f(\xi, y_{n-1}, \varepsilon)}{\partial y} \frac{\partial y_n}{\partial \varepsilon} + \frac{\partial f(\xi, y_{n-1}, \varepsilon)}{\partial \varepsilon} \right) d\xi, \]
\[ y_n(0) = y_0, \quad y_0 = \varphi(x). \]
The advantages of replacing the problem (5), (6) with the problem (7) are obvious: the new argument \( \varepsilon \) ranges over the small interval \( 0 \leq \varepsilon \leq \varepsilon_0 \), while the previous argument \( x \) ranged over a half-line; and it is well-known that approximate methods converge faster in small intervals than in large ones. We note that the proposed scheme has long been applied by physicists (see, for example, [3]) under the name of the ECC-method (evolution of the coupling constant) in quantum mechanics. A brief historical account of the development of this method can be found in [3]. A. A. Dorodnitsyn later applied the method for numerical solution of differential equations [2].

In the present paper we justify the method sketched above and pose some new problems that are of use in both pure and applied mathematics. We also include one reflection of a natural-philosophy character, which will inspire research and help to preserve confidence in success amid the thicket of the mathematical computations. The structure of our planetary system is not uniquely determined. So-called mutually attracting multiple-star systems are known in astronomy, and this phenomenon is regarded as typical. Each galaxy likewise is a collection of stars that are attracted to the center and interact with one another. In addition galactic clusters are known, which form mutually gravitating systems. It seems, in brief, that there exists in the cosmos an immense number of systems analogous in a certain sense to the solar system; therefore there must exist effective methods of computing the motion of these systems. We remark that the rapid method of solving systems of differential equations with a small parameter is one such method. The question naturally arises of extending this method to systems of partial differential equations and systems of integro-differential equations and subsequent generalization to equations in normed spaces.

An outline of the method and its justification. Consider a system of ordinary differential equations with initial conditions on the unknown functions
\[ \frac{dy_i}{dx} = f_i(x, y_1, \ldots, y_n, \varepsilon), \quad i = 1, n, \]
\[ y_i|_{x=x_0} = y_{i0}, \quad i = 1, n, \]
where the functions \( f_i \), which are defined in a certain real domain
\[ D = \{ |y_i - y_{i0}| \leq a_i, \quad 1 \leq i \leq n, \quad x_0 \leq x \leq a, \quad 0 \leq \varepsilon \leq 1 \}, \]
and have continuous derivatives up to order three in all their arguments, and \( \varepsilon \) is a small parameter. Suppose the problem (8), (9) has a solution that is stable with respect to \( \varepsilon \), that in the domain \( D \) there exist continuous derivatives of the functions \( y_i \) with respect to the small parameter \( \varepsilon \), \( i = 1, n \), and \( y_i|_{\varepsilon=0} = \varphi_{i0}(x) \), where \( \varphi_{i0}(x) \) are functions that are continuously differentiable in their domain of definition, and \( \varphi_{i0}(x_0) = y_{i0}, \quad i = 1, n \).