THE ELEMENT SETS AND VALUE SETS
OF TWO-DIMENSIONAL CONTINUED FRACTIONS

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For two-dimensional continued fractions we prove the existence and uniqueness of an optimal sequence of value sets corresponding to an arbitrarily given sequence of element sets. We compute the element set for a given sequence of disk value sets and, as a corollary, give the element sets and value sets that are used in convergence criteria for two-dimensional continued fractions.

The known convergence criteria for multidimensional continued fractions are predominantly determined by the type of region of convergence. This leads to consideration of the element sets and value sets of multidimensional continued fractions.

Consider a sequence of nonempty sets \( \{\Omega_{ij}\} \) and \( \{\mathfrak{V}_{ij}\} \), \( \Omega_{ij} \subset \mathbb{C} \times \mathbb{C} \), \( \mathfrak{V}_{ij} \subset \mathbb{C} \) and the two-dimensional continued fraction (1)

\[
\begin{align*}
& \frac{a_{00}}{b_{00} + \sum_{i=1}^{\infty} \frac{a_{i0}}{b_{i0}}} + \frac{a_{11}}{b_{11} + \sum_{i=2}^{\infty} \frac{a_{i1}}{b_{i1}}} = \frac{a_i}{b_i + \sum_{j=1}^{\infty} \frac{a_{ij}}{b_{ij}}}, \\
& i = 0, 1, \ldots
\end{align*}
\]

We introduce the following recursion relations

\[
Q^{(s)}_{1,s} = b_{k+s,k}, \quad Q^{(s)}_{1,j} = b_{k+j,j} + \frac{a_{k+j+1,k}}{Q^{(s)}_{1,j+1}}, \quad Q^{(s)}_{2,s} = b_{k,k+s} + \frac{a_{k,k+s+1}}{Q^{(s)}_{2,j+1}}, \quad Q^{(s)}_{2,j} = b_{k+j,j} + \frac{a_{k+j,1}}{Q^{(s)}_{2,j+1}}.
\]

by means of which we can write the \( m \)th approximant of the two-dimensional continued fraction (1) or any link of it in the form

\[
\begin{align*}
& \frac{a_{kk}}{b_{kk} + \frac{a_{k+1,k}}{Q^{(m-2k-1)}_{1,k}} + \frac{a_{k+1,k+1}}{Q^{(m-2k-3)}_{1,k+1}}}, \\
& k = 0, 1, \ldots
\end{align*}
\]

For the two-dimensional continued fraction (1) the sequence \( \{\Omega_{ij}\} \) is called a sequence of element sets and \( \{\mathfrak{V}_{ij}\} \) a sequence of value sets corresponding to \( \{\Omega_{ij}\} \) if

\[
\begin{align*}
& t_{ij}(v_{i+1,j}) = \frac{a_{ij}}{b_{ij} + v_{i+1,j}} \subset \mathfrak{V}_{ij} \quad (i > j), \quad t_{ij}(v_{i,j+1}) = \frac{a_{ij}}{b_{ij} + v_{i,j+1}} \subset \mathfrak{V}_{ij} \quad (i < j), \\
& t_{ii}(v_{i+1,i}, v_{i,i+1}, v_{i+1,i+1}) = \frac{a_{ii}}{b_{ii} + v_{i+1,i} + v_{i,i+1} + v_{i+1,i+1}} \subset \mathfrak{V}_{ii}
\end{align*}
\]

for arbitrary \((a_{ij}, b_{ij}) \in \Omega_{ij}, v_{ij} \in \mathfrak{V}_{ij}, i, j = 0, 1, \ldots\).
If \( \{V_{nm}\} \) is a sequence of value sets corresponding to the sequence of element sets \( \{\Omega_{nm}\} \), and if for each sequence of value sets \( \{V_{nm}\} \) corresponding to \( \{\Omega_{nm}\} \) we have \( V_{nm} \subseteq V_{nm}^1 (n, m = 0, 1, \ldots) \), the sequence \( \{V_{nm}\} \) is called the optimal sequence of value sets corresponding to the sequence of element sets \( \{\Omega_{nm}\} \).

Similarly one can speak of an optimal sequence of element sets \( \{\Omega_{nm}\} \). The existence and uniqueness of the optimal sequence of value sets corresponding to an arbitrarily given sequence of element sets is proved by the following theorem.

**Theorem 1.** Let \( \{\Omega_{ij}\} \) be a sequence of element sets and \( \{V_{ij}\} \) a corresponding sequence of value sets. Then

1) for an arbitrary \( m \geq 2k + 1, \ k = 0, 1, \ldots \), we have

\[
\begin{align*}
\frac{a_{kk}}{Q_k^{(m-2k-1)}} & \in V_{kk}, \\
\frac{a_{nk}}{Q_k^{(m-2k-1)}} & \in V_{nk}, \\
\frac{a_{kn}}{Q_{2,n-k}^{(m-2k-1)}} & \in V_{nk}, \\
& m - 1 > n > k, \quad n = 1, 2, \ldots,
\end{align*}
\]

if \((a_{ij}, b_{ij}) \in \Omega_{ij}, k \leq i \leq m - 1, k \leq j \leq m - 2k - 1;\)

2) the sequence

\[
W_{nk} = \begin{cases} \\
\frac{a_{kk}}{Q_k^{(m-2k-1)}} : (a_{ij}, b_{ij}) \in \Omega_{ij}, k \leq i \leq j \leq m - 2k - 1, \text{ if } n = k, \\
\frac{a_{nk}}{Q_{1,n-k}^{(m-2k-1)}} : (a_{pk}, b_{pk}) \in \Omega_{pk}, k < p < 2m - 2k - 1, \text{ if } n > k, \\
\frac{a_{kn}}{Q_{2,n-k}^{(m-2k-1)}} : (a_{kp}, b_{kp}) \in \Omega_{kp}, k < p < 2n - 2k - 1, \text{ if } n < k,
\end{cases}
\]

is the optimal sequence of value sets corresponding to \( \{\Omega_{nk}\} \).

**Proof.** When \( m = n + k + 1 \) we have

\[
Q_{1,n-k}^{(n-k)} = b_{nk}, \quad Q_{2,n-k}^{(n-k)} = b_{kn}
\]

and assertion (6) holds, since \( \frac{a_{nk}}{b_{nk}} \in V_{nk} \) and \( \frac{a_{kn}}{b_{kn}} \in V_{kn} \).

To prove assertions (6) we apply complete induction with respect to \( s = m - k - 1 - n \). Let

\[
\frac{a_{pk}}{Q_{1,p-k}^{(r-2k-1)}} \in V_{pk}, \quad \frac{a_{kp}}{Q_{2,p-k}^{(r-2k-1)}} \in V_{kp}
\]

for arbitrary sets of indices such that \( r - p - k - 1 \leq s \). Then

\[
\frac{a_{ik}}{Q_{1,i-k}^{(r-2k-1)}} \in V_{ki}, \quad \frac{a_{ki}}{Q_{2,i-k}^{(r-2k-1)}} \in V_{ki},
\]

where the indices are arbitrary and \( t - i - k \leq s + 1 \).

Indeed, in accordance with the notation (2), we have

\[
Q_{1,j}^{(m-2k-1)} = b_{k+j,k} + \frac{a_{k+j+1,k}}{Q_{1,j+1}^{(m-2k-1)}}, \quad Q_{2,j}^{(m-2k-1)} = b_{k,k+j} + \frac{a_{k,k+j+1}}{Q_{2,j+1}^{(m-2k-1)}}, \quad j = 1, \ldots, m - 2k - 1,
\]

where

\[
Q_{1,m-2k-1}^{(m-2k-1)} = b_{m-k-1,k}, \quad Q_{m-2k-1}^{(m-2k-1)} = b_{k,m-k-1}, \quad \frac{a_{ik}}{Q_{1,j-k}^{(r-2k-1)}} = \frac{a_{ik}}{b_{ik} + \frac{a_{ik}}{Q_{1,j-k}^{(r-2k-1)}}} \in V_{i,k}.
\]

2369