APPROXIMATION OF THE LAURICELLA HYPERGEOMETRIC FUNCTIONS $F^{(N)}_D$ BY BRANCHED CONTINUED FRACTIONS

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Applying recursion relations for the Lauricella hypergeometric functions $F^{(N)}_D$, we construct an expansion of a ratio of these functions in branched continued fractions. We study the convergence of the resulting expansion in the case of real parameters.

Lauricella [3] defined and partly studied hypergeometric functions of $N$ variables

\[
F^{(N)}(a_1, \ldots, a_N; b_1, \ldots, b_N; c_1, \ldots, c_N; z_1, \ldots, z_N) = \sum_{k_1, \ldots, k_N=0}^{\infty} (a_1)^{k_1} \cdots (a_N)^{k_N} (b_1)^{k_1} \cdots (b_N)^{k_N} z_1^{k_1} \cdots z_N^{k_N} \]

where $a_1, \ldots, a_N, b_1, \ldots, b_N, c_1, \ldots, c_N$ are complex constants with $c_1, \ldots, c_N \neq 0, -1, -2, \ldots,$ and $z_1, \ldots, z_N$ are complex variables. We denote the Pochhammer symbol by $(a)^{k} = (a)(a+1)\cdots(a+k-1)$, and $(a)^{0} = 1$.

The subject of the present study is the Lauricella hypergeometric function $F^{(N)}_D$.

The following recursion relations hold for the hypergeometric functions $F^{(N)}_D(a, \tilde{b}; c; \tilde{z})$

\[
F^{(N)}_D(a, \tilde{b}; c; \tilde{z}) = F^{(N)}_D(a, \tilde{b} + \delta_1; c; \tilde{z}) - \frac{\tilde{\alpha} \tilde{z}_1}{c} F^{(N)}_D(a + 1, \tilde{b} + \delta_1; c + 1; \tilde{z}), \quad i = 1, N, \quad (2)
\]

\[
F^{(N)}_D(a, \tilde{b}; c; \tilde{z}) = \frac{c - \tilde{\alpha}}{c} F^{(N)}_D(a, \tilde{b}; c + 1; \tilde{z}) + \frac{\tilde{\alpha}}{c} F^{(N)}_D(a + 1, \tilde{b}; c + 1; \tilde{z}), \quad (3)
\]

\[
F^{(N)}_D(a, \tilde{b}; c; \tilde{z}) = F^{(N)}_D(a + 1, \tilde{b}; c; \tilde{z}) - \sum_{i=1}^{N} \frac{b_i}{c} z_i F^{(N)}_D(a + 1, \tilde{b} + \delta_i; c + 1; \tilde{z}), \quad (4)
\]

where $\tilde{b} = (b_1, b_2, \ldots, b_N)$, $\tilde{z} = (z_1, z_2, \ldots, z_N)$, $\delta_i = (\delta^1_i, \delta^2_i, \ldots, \delta^N_i)$, and $\delta^j_i$ is the Kronecker symbol. Formulas (3) and (4) were given without proof in the monograph [2].

We shall construct an expansion of a ratio of two Lauricella hypergeometric functions

\[
F = \frac{F^{(N)}_D(a + 1, \tilde{b} + \delta_1; c + 1; \tilde{z})}{F^{(N)}_D(a, \tilde{b}; c; \tilde{z})} \quad (5)
\]

in a branched continued fraction. It follows from relation (2) that

\[
\frac{F^{(N)}_D(a, \tilde{b}; c; \tilde{z})}{F^{(N)}_D(a + 1, \tilde{b} + \delta_1; c + 1; \tilde{z})} = \frac{F^{(N)}_D(a, \tilde{b} + \delta_1; c; \tilde{z})}{F^{(N)}_D(a + 1, \tilde{b} + \delta_1; c + 1; \tilde{z})} - \frac{\tilde{\alpha}}{c} z_1 . \quad (6)
\]


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Taking account of formula (3), we obtain

\[
\frac{F_D^{(N)}(a, b + \bar{e}_i; c; \bar{z})}{F_D^{(N)}(a + 1, b + \bar{e}_i; c + 1; \bar{z})} = \frac{c - a}{c} \frac{F_D^{(N)}(a, b + \bar{e}_i; c + 1; \bar{z})}{F_D^{(N)}(a + 1, b + \bar{e}_i; c + 1; \bar{z})} + \frac{a}{c}. \tag{7}
\]

It follows from the property (4) that

\[
\frac{F_D^{(N)}(a, b + \bar{e}_i; c + 1; \bar{z})}{F_D^{(N)}(a + 1, b + \bar{e}_i; c + 1; \bar{z})} = \frac{1}{1 + \sum_{k=1}^{N} \frac{b_k + \delta_k}{c + 1} \frac{F_D^{(N)}(a, b + \bar{e}_i; c + 1; \bar{z})}{F_D^{(N)}(a, b; c; \bar{z})}}. \tag{8}
\]

We introduce the notation

\[
G_i(a, \tilde{b}; c; \bar{z}) = \frac{F_D^{(N)}(a + 1, b + \bar{e}_i; c + 1; \bar{z})}{F_D^{(N)}(a, b; c; \bar{z})}, \quad 1 \leq i \leq N.
\]

Taking account of relations (6)-(8), we have

\[
G_i(a, \tilde{b}; c; \bar{z}) = \frac{1}{\frac{a}{c}(1 - z_i) + \frac{1 - a}{c}} \frac{1}{1 + \sum_{k=1}^{N} \frac{b_k + \delta_k}{c + 1} z_k G_k(a, \tilde{b} + \bar{e}_i; c + 1; \bar{z})}, \tag{9}
\]

where \( i = 1, 2, \ldots, N \). By successively combining relations (9), we obtain the following theorem.

**Theorem 1.** The ratio of two Lauricella hypergeometric functions has a formal expansion in a branched continued fraction

\[
\frac{F_D^{(N)}(a + 1, \tilde{b} + \bar{e}_i; c + 1; \bar{z})}{F_D^{(N)}(a, \tilde{b}; c; \bar{z})} = G_1(a, \tilde{b}; c; \bar{z}) = \frac{1}{u_0(\bar{z}) + \sum_{i=1}^{N} \frac{u_i(1)(\bar{z})}{u_i(0)(\bar{z}) + \sum_{i=1}^{N} \frac{u_i(n)(\bar{z})}{u_i(n-1)(\bar{z}) + \sum_{i=1}^{N} \frac{u_i(n-1)(\bar{z})}{\ldots}}}} \tag{10}
\]

where

\[
u_0(\bar{z}) = \frac{a}{c}(1 - z_1), \quad w_n = 1 - \frac{a}{c + n - 1}, \quad u_i(n) = \frac{b_i + p_i(n) z_i}{c + n},
\]

\[
u_i(n) = \frac{a}{c + n}(1 - z_i), \quad p_i(n) = \begin{cases} \alpha_i(n), & \text{if } i_n \neq 1, \\ \alpha_i(n) + 1, & \text{if } i_n = 1; \end{cases}
\]

\( \alpha_i(n) \) is the number of occurrences of \( i_n \) in the multi-index \( i(n - 1) = i_1 i_2 \ldots i_{n-1} \) if \( n \geq 2 \), and \( \alpha_i(1) = 0 \).