THE SUBGROUPS OF THE SPECIAL LINEAR GROUP OVER A SKEW FIELD THAT CONTAIN THE GROUP OF DIAGONAL MATRICES

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For any (noncommutative) skew field $T$, the lattice of subgroups of the special linear group $\Gamma = SL(n,T)$ that contain the subgroup $\Delta = SD(n,T)$ of diagonal matrices (with Dieudonné determinants equal to 1) is studied. It is established that for any subgroup $H$, $\Delta \leq H \leq \Gamma$, there exists a uniquely determined unital net $\sigma$ such that $\Gamma(\sigma) \leq H \leq N(\sigma)$, where $\Gamma(\sigma)$ is the net subgroup associated with the net $\sigma$ and $N(\sigma)$ is its normalizer in $\Gamma$. Bibliography: 11 titles.

§1. INTRODUCTION

The problem concerning the standard description of subgroups of the special linear group $SL(n,k)$, $n \geq 2$, over a field $k$ that contain the group $SD(n,k)$ of diagonal matrices was studied by several authors. Seitz [9] considered the case $n \geq 2$, $k = F_q$ ($F_q$ is the finite field with $q$ elements), where $q \geq 13$, $q \neq 2^m$. The case $n = 2$, $k$ infinite, char($k$) $\neq 2$, and Card($k^*/k^*$) $\leq$ Card($k^*$) was studied by Vavilov and Dybkova (see [10]). Later, in a series of articles (see [5]) Vavilov gave the standard description of these subgroups for the case $n \geq 3$ and $k$ an arbitrary field with the condition that Card($k$) $\geq 7$. In [11], there are several examples from which it follows that the condition Card($k$) $\geq 7$ is necessary. Thus, the result of Vavilov in [5] is best possible for an arbitrary field $k$. In this article we continue our study of an analogous problem for the case of skew fields begun in [4], where we gave the standard description of these subgroups in the special linear group over a skew field with infinite center. Here we shall discuss the case of a skew field with an arbitrary center that contains no less than seven elements. We make use of the following notation in our article: $\Lambda$ is an associative ring with 1, $\Lambda^*$ is the group of invertible elements of $\Lambda$; $G = GL(n, \Lambda)$ is the general linear group of degree $n$ over $\Lambda$; $D = D(n, \Lambda)$ is the subgroup of diagonal matrices in $G$; $e = e_n$ is the identity matrix of degree $n$; $e$ is the matrix having 1 at the position $(i,j)$ and zero at all of the remaining positions; $t_{ij}(\alpha)$ is the elementary transvection $e + \alpha e_{ij}$, $\alpha \in \Lambda(i \pm j)$; $d_r(e)$ is the diagonal matrix $e + (e - 1)e_{rr}$ ($e \in \Lambda^*$, $1 \leq r \leq n$); $T$ is a skew field; $z$ is the center of $T$; $[T^*, T^*]$ is the commutator subgroup of the multiplicative group $T^*$ of $T$; $\Gamma = SL(n,T)$ is the special linear group of degree $n$; $\Delta = SD(n,T)$ is the subgroup of diagonal matrices the Dieudonné determinants of which are equal to 1; $d_{rs}(\varepsilon)$ is the diagonal matrix $e + (e - 1)e_{rr} + (e - 1) e_{ss}$ for $r \neq s$ and $\varepsilon \in T^*$. If $a = (a_{ij})$ is an invertible matrix, then we denote $a^{-1} = (a_{ij}^{-1})$.

§2. NETS AND NET SUBGROUPS

Let $\Lambda$ be an arbitrary associative ring with identity 1. For a natural number $n$ we consider a square array

$$\sigma = (\sigma_{ij}), \quad 1 \leq i, j \leq n,$$

where all $\sigma_{ij}$ are two-sided ideals of $\Lambda$. This array is called a net of degree $n$ of ideals in $\Lambda$ (in brief, a net of degree $n$ over $\Lambda$) if

$$\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij} \quad (1)$$

for all values of the indices $i, j$, and $r$.

A net $\sigma$ is called a unital net if $\sigma_{ii} = \Lambda$ for all $i = 1, 2, \ldots, n$. For a given net $\sigma$ we denote by $M(\sigma)$ the collection of all matrices $a = (a_{ij})$ in the ring $M(n, \Lambda)$ of matrices of degree $n$ over $\Lambda$ for which $a_{ij} \in \sigma_{ij}$ for all $i$ and $j$. In addition, we consider the following concept, which was introduced in [2].

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Definition. Let $\sigma$ be an arbitrary net of degree $n$ over a ring $A$. The largest subgroup of the general linear group $G = GL(n, A)$ contained in the multiplicative system $e + M(\sigma)$ is called the net subgroup in $GL(n, A)$ that corresponds to the net $\sigma$ and is denoted by $G(\sigma)$. If $\sigma$ is a unital net, then $G(\sigma)$ is also called a unital net subgroup.

Let $R$ be a commutative ring with $1$. The subgroup of the group $GL(n, R)$ that consists of all invertible matrices with determinant equal to 1 is called the special linear group of degree $n$ over $R$. This group is denoted by $F = SL(n, R)$.

Let $A$ be a commutative ring with identity $1$. We consider a subgroup $H, D(n, A) \leq H \leq GL(n, A)$. If $A$ is generated by its invertible elements as a ring, i.e., if each element of $A$ is represented as a sum of invertible elements, then the subgroup $H$ corresponds to some unital net of degree $n$ over $A$. In fact, if $i \neq j$ we put

$$\sigma_{ij} = \alpha \in A/t_{ij}(\alpha) \in H.$$  

Then $\sigma_{ij}$ is a two-sided ideal in view of the following obvious formulas:

$$d_i(\varepsilon)t_{ij}(\alpha)d_i(\varepsilon^{-1}) = t_{ij}(\varepsilon\alpha), \quad \varepsilon \in \Lambda^*,$$

$$d_j(\varepsilon)t_{ij}(\alpha)d_j(\varepsilon^{-1}) = t_{ij}(\varepsilon\alpha), \quad \varepsilon \in \Lambda^*.$$

If, moreover, we put $\sigma_{ii} = \Lambda$ for all $i = 1, 2, \ldots, n$, we shall have a unital net $\sigma = (\sigma_{ij})$. This net is called the unital net associated with the subgroup $H$.

Now let $A$ be a semilocal ring and let $F = SL(n, A)$ be the special linear group over $A$. Consider a subgroup $H, ED(n, A) \leq H \leq SL(n, A)$, where $ED(n, A) = E(n, A) \cap D(n, A)$. This subgroup $H$ is called an intermediate subgroup. Put

$$\sigma_{ij} = \alpha \in A/t_{ij}(\alpha) \in H, \quad i \neq j.$$

Then $\sigma$ is a two-sided ideal in view of the following formulas:

$$t_{ij}(\varepsilon\alpha) = d_k(\varepsilon)t_{ij}(\alpha)d_k(\varepsilon^{-1}), \quad \varepsilon \in \Lambda^*, \quad k \neq i,$$

$$t_{ij}(\varepsilon\alpha) = d_k(\varepsilon)t_{ij}(\alpha)d_k(\varepsilon^{-1}), \quad \varepsilon \in \Lambda^*, \quad k \neq j.$$

Moreover, for all $i = 1, 2, \ldots, n$ we put $\sigma_{ii} = \Lambda$. Then the array $\sigma = (\sigma_{ij})$ is the unital net associated with the subgroup $H$.

Now, let $T$ be a skew field, $\Gamma = SL(n, T)$ the special linear group over $T$, and $\Delta = SD(n, T)$ the subgroup of diagonal matrices whose Dieudonné determinants are equal to 1. Let $Z$ be the center of $T$. If $Z$ is infinite, then for all $n \geq 3$ the standard description of the intermediate subgroups was obtained in [4]. In fact, the following theorem was proved.

Theorem 1 (see [4]). Let $T$ be a skew field with infinite center and $n \geq 3$. Assume that $\Gamma = SL(n, T)$ and $\Delta = SD(n, T)$. Then for each subgroup $H, \Delta \leq H \leq \Gamma$, there exists a unique unital net $\sigma$ of degree $n$ over $T$ such that

$$\Gamma(\sigma) \leq H \leq N(\sigma),$$

where $\Gamma(\sigma)$ is the net subgroup of $\Gamma$ that corresponds to the net $\sigma$ and $N(\sigma)$ is the normalizer of the subgroup $\Gamma(\sigma)$ in $\Gamma$.

In this article, we consider the case of a skew field $T$ with a weaker condition that the center of $T$ contains no less than seven elements.