ARITHMETIC EMBEDDINGS OF ALGEBRAIC GROUPS

M. N. Kubenskii

Embeddings of algebraic groups are considered. The action of an algebraic group on these embeddings preserving their arithmetic properties is defined. A formula for the number of orbits is presented. Bibliography 3 titles.

Let $U$ and $V$ be two linear spaces over the field of rational numbers $\mathbb{Q}$, and let algebraic groups $H$ and $G$ act on $U$ and $V$, respectively. We fix a lattice $\Gamma$ in $V$ and a lattice $\Lambda$ in $U$. We consider an embedding

$$\rho_0 : U \rightarrow V$$

with "arithmeticity" condition

$$\rho_0(\Lambda) = \rho_0(U) \cap \Gamma.$$  

As usual (see [2]), with every global object we associate its adele analog, which we provide with the letter $A$ as a subscript, e.g., $G$ and $G_A$, $V$ and $V_A$, etc. We note that embeddings of global objects give rise to embeddings of adele objects.

We consider the embeddings of $U$ in $V$ that have the form $g \cdot \rho_0$ with $g \in G$. With an embedding of the linear spaces $\rho : U \rightarrow V$ we associate an embedding of the algebraic groups $\tau : H \rightarrow G$ defined by the relation

$$\tau(h)\rho(u) = \rho(hu).$$

We denote by $\tau_0$ the embedding of groups related to the embedding $\rho_0$. Then the embedding of groups $g\tau_0g^{-1}$ is associated with the embedding $g \cdot \rho_0$. The natural action of the group $G_A$ on the embeddings $\{g\rho_0\}$ does not preserve the arithmeticity condition, and we define another action, which is more closely related to the "arithmeticity" condition.

Let $G_{\Gamma'}$ denote the group of elements of $G$, not changing $\Gamma'$ (the so-called integral points of an algebraic group (see [2])). Two lattice $\Gamma'$ and $\Gamma''$ are called equivalent if there exists an element $g \in G$ such that $g(\Gamma') = \Gamma''$. Two lattices $\Gamma'$ and $\Gamma''$ belong to one and the same genus if there exists an element $g_A \in G_A$ such that $g_A(\Gamma') = \Gamma''$. Let $\Gamma_1, \ldots, \Gamma_q$ be representatives of the genus classes of the lattice $\Gamma$. Let $\Gamma_i = \psi_i \Gamma$, where $\psi_i \in G_A$. Then $G_A = \bigcup_{i=1}^q G\psi_i G_{\Gamma_i,A}$. It is clear that $G_{\Gamma_i,A} = \psi_i G_{\Gamma_i,A} \psi_i^{-1}$, $G_A = \bigcup_{i=1}^q G\psi_r \psi_i^{-1} G_{\Gamma_i,A}$.

Let $p \in G$, and let $pp_0(\Lambda) = pp_0(U) \cap \Gamma$. For any $g_A \in G_A$ we have

$$pg_{AP}^{-1} = g\psi_j \psi_i^{-1} e_A^{(i)} \in \bigcup_{r=1}^q G\psi_r \psi_i^{-1} G_{\Gamma_i,A}.$$  

(1)

We define the action of $g_A$ on $pp_0$ in the following way:

$$g_A[pp_0] = g^{-1} \cdot pp.$$  

We show that this action is coordinated with the group structure. Let $g_A = g_A^{(1)} \cdot g_A^{(2)}$,

$$pg_{A}^{(2)}p^{-1} = g_2 \psi_j \psi_i^{-1} e_A^{(i)},$$  

$$g_2^{-1} pg_{A}^{(1)}p^{-1} g_2 = g_1 \psi_r \psi_j^{-1} e_A^{(j)}.$$  


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Then
\[ p g_A^{-1} = g_2 g_1 \psi_r \psi_j^{-1} e_A^{(i)} g_2^{-1} g_1 \psi_r \psi_j^{-1} e_A^{(i)} = g_2 g_1 \psi_r \psi_j^{-1} ((\psi_r \psi_j^{-1} e_A^{(i)}) \psi_j \psi_r^{-1}) e_A^{(i)} = g_2 g_1 \psi_r \psi_j^{-1} e_A^{(i)}. \]

Hence,
\[ g_A^{(1)} [g_A^{(2)} p_{p0}] = g_A^{(1)} [g_2^{-1} p_{p0}] = g_1^{-1} g_2^{-1} p_{p0} = (g_2 g_1)^{-1} p_{p0} = g_A [p_{p0}]. \]

It should be mentioned that the action introduced above is not well defined because decomposition (1) is not unique. However, as in [1], we may show that it becomes well defined if instead of the mapping $p_{p0}$, we define the action on the classes of embeddings $p_{p0}$ formed by the embeddings which differ by elements from $G_{\Gamma_1}$: $p_{p0} = \{e_{\Gamma_1}, p_{p0}\}$, where $e_{\Gamma_1} \in G_{\Gamma_1}$. We note that a class of embeddings of linear spaces corresponds to a class of embeddings of an algebraic group $H$ in an algebraic group $G$. Let $\mathcal{M}(\Lambda, \Gamma_i)$ denote the set of embeddings $\rho$ from $\{G_{\rho_0}\}$ that have the property $\rho(\Lambda) = \rho(U) \cap \Gamma_i$. Accordingly, we denote by $\mathcal{M}(\Lambda, \Gamma_i)$ their equivalence classes with respect to $G_{\Gamma_1}$. We note that if $\rho \in \mathcal{M}(\Lambda, \Gamma_i)$ and $\rho \in \mathcal{M}(\Lambda, \Gamma_j)$, then we shall assume that these embeddings are distinct.

We denote by $T_{A_i}^{(1)}$ the set of elements $t_A \in G_A$ with the following property: if $\rho \in \mathcal{M}(\Lambda, \Gamma_i)$, then $t_A \rho(\Lambda) = \rho(U) \cap \Gamma_i$. Let $g_A^{-1} \in T_{A_i}^{(1)}$, $\rho \in \mathcal{M}(\Lambda, \Gamma_i)$, and $g_A = g_\psi \psi^{-1} e_A^{(i)}$. Then $g_A[\rho] \in \mathcal{M}(\Lambda, \Gamma_i)$.

Indeed, $g_A[\rho] = g^{-1} \cdot \rho$. Since $g_A^{-1} \in T_{A_i}^{(1)}$, we have
\[ e_A^{(i)} \psi_j \psi_j^{-1} g^{-1} \rho(\Lambda) = e_A^{(i)} \psi_j \psi_j^{-1} g^{-1} \rho(U) \cap \Gamma_i, \]
and, therefore,
\[ g^{-1} \rho(\Lambda) = g^{-1} \rho(U) \cap \psi_1 \psi_1^{-1} e_A^{(i)} \Gamma_i = g^{-1} \rho(U) \cap \Gamma_j. \]

Now we study under what conditions we have
\[ g_A^{(1)}[\rho] \sim g_A^{(2)}[\rho], \quad \text{i.e.,} \quad g_A^{(1)}[\rho] = g_A^{(2)}[\rho]. \]

Let $\rho \in \mathcal{M}(\Lambda, \Gamma_i)$. Then $g_A^{(1)} = g_1 \psi \psi^{-1} e_A^{(i)}$, $g_A^{(2)} = g_2 \psi \psi^{-1} e_A^{(i)}$. We denote by $E_\rho$ the set of elements of $G$ that do not change $\rho$. We have $g_2^{-1} \rho = e_{\Gamma_1} g_1^{-1} \rho$. Hence, $g_2 e_{\Gamma_1} g_1^{-1} \in E_\rho$ or $g_2 \in E_\rho g_1 e_{\Gamma_1}^{-1}$. Multiplying by suitable factors, we have
\[ g_A^{(2)} \in E_\rho g_1 e_{\Gamma_1}^{-1} \psi \psi^{-1} e_A^{(i)} \psi_j \psi_j^{-1} e_A^{(i)} = E_\rho g_1 \psi \psi^{-1} e_A^{(i)} \cdot e_A^{(i)} \psi_j \psi_j^{-1} e_{\Gamma_1}^{-1} \psi_j \psi_j^{-1} e_A^{(i)} e_{\Gamma_1}^{-1} e_A^{(i)} e_{\Gamma_1}^{-1} \in E_\rho g_A^{(1)} G_{\Gamma_i, A}. \]

Obviously, the converse is also valid: if $g_A^{(2)} \in E_\rho g_A^{(1)} G_{\Gamma_i, A}$, then $g_A^{(1)}[\rho] = g_A^{(2)}[\rho]$.

Now we turn to the algebraic groups $H$ and $G$. Let us consider the embedding $\tau_0 : H \to G$ related to the embedding $p_0 : U \to V$. Let $H^c$ designate the group that is isomorphic to the subgroup of elements of $C$ commuting with the elements from $\tau_0(H)$. We denote the corresponding embedding of this group by $\tau_0 : H^c \to G$. Obviously, if $p_0$ is replaced by $g \cdot p_0$, then $\tau_0$ is replaced by $g \tau_0 g^{-1}$ and $\tau_0$ by $g \tau_0 g^{-1}$. Hence, to every embedding $\rho : U \to V$ there correspond two embeddings $\tau : H \to G$ and $\tau : H^c \to G$. If $\rho \in \mathcal{M}(\Lambda, \Gamma_i)$, then $\tau(H^c) \subset T^{(i)} = T_{A_i}^{(i)} \cap G$, $\tilde{\tau}^t(H^c) \subset T^{(i)}$. Now we define the action of the group $H^c_A$ on \( \bigcup_{i=1}^{q} \mathcal{M}(\Lambda, \Gamma_i). \) Let $h^*_A \in H^c_A$ and $\bar{\rho} \in \mathcal{M}(\Lambda, \Gamma_i)$. We choose $\rho \in \bar{\rho}$ and $\tilde{\tau}$, the embedding of $H^c_A$ in $G_A$, associated with $\rho$. Then $\tilde{\tau}(h^*_A)^{-1} \in T^{(i)}$, whence
\[ \tilde{\tau}(h^*_A)[\rho] \in \bigcup_{i=1}^{q} \mathcal{M}(\Lambda, \Gamma_i). \]

We define
\[ h^*_A[\rho] = \tilde{\tau}(h^*_A)[\rho] \in \bigcup_{i=1}^{q} \mathcal{M}(\Lambda, \Gamma_i). \]

Taking into account condition (2) under which the relation $g_A^{(1)}[\rho] = g_A^{(2)}[\rho]$ is valid, we have the following theorem.