It is proved that the group of spinor norms of automorphies of a generalized quadratic lattice \( L \) over the ring of integral elements of a local field \( \mathbb{Z}_p \), in the case where \( p \nmid 2 \) and \( L \) is a generalized translation, is generated by the spinor norms of symmetries contained in the group of automorphies of \( L \). As a corollary, an extension to the case of generalized quadratic lattices is given for known sufficient conditions of coincidence of the genus and the spinor genus of a quadratic lattice. Bibliography: 9 titles.

§1. INTRODUCTION

The notion of spinor genus introduced by Eichler [5] is one of the central notions in the modern theory of quadratic forms over algebraic number fields. It turns out that the main properties of spinor genera may be described in terms of the groups \( \theta(O^+(L_p)) \), i.e., the spinor norms of local automorphies for the lattice \( L \) corresponding to a quadratic form. For odd valuations \( p \nmid 2 \) these groups were calculated by Kneser [7], and for even \( p \mid 2 \) the same was done for the most important cases by Hsia and Earnest [6, 4]. It appears that in all cases investigated, these groups are generated by the spinor norms of symmetries belonging to \( O(L_p) \) (more exactly, by pairs of such symmetries). Moreover, essentially in all cases the group of automorphies \( O(L_p) \) itself is generated by such symmetries.

In [2], which is being prepared for publication, we introduce the notion of generalized quadratic lattices. This notion proves to be very useful when studying the representations of forms by forms, in particular, when studying the representations with additional congruence conditions. The main results of the arithmetic theory of quadratic forms are carried over to generalized lattices. In particular, the classes, genera, and spinor genera of generalized quadratic lattices are defined, and their study reduces to calculation of the groups \( \theta(O^+(L_p)) \), the spinor norms of local automorphies but for generalized quadratic lattices.

In the general case, the computation of these groups is a very difficult problem. However, in solving the great majority of problems of the classical theory of quadratic forms it proves to be sufficient to restrict oneself to generalized lattices of a special kind, the so-called generalized translations. Our main purpose in this note is to compute the group \( \theta(O^+(L_p)) \) for a generalized translation \( L \) and for odd \( p \mid 2 \). We show that (see Theorem 2) in this case, as for classical quadratic lattices, the group \( \theta(O(L_p)) \) is generated by the spinor norms of the symmetries \( \tau \in O(L_p) \) and \( \theta(O^+(L_p)) \) by the products of pairs of the spinor norms of such symmetries. In addition, the group of local automorphies \( O(L_p) \) of a generalized quadratic lattice as opposed to the classical case is no longer generated by the symmetries that are contained in it even for odd \( p \mid 2 \); in fact, for a generalized translation \( L \) we must add to them the products of two or even four symmetries of special kind, but these additional automorphies are of spinor norm 1.

As a consequence, we present without proof an extension of a known result that, if the discriminant of a quadratic \( \mathbb{Z} \)-lattice does not contain large powers of prime numbers, then the genus of such a lattice coincides with the spinor genus, and also further extension of this result to the case of algebraic number fields (Theorem 3 and its corollaries). In particular, for indefinite quadratic forms this yields a sufficient condition of the fact that the genus of generalized lattices consists of one class.

§2. MAIN DEFINITIONS AND RESULTS

We try to follow the notation of the books of O'Meara [9] and Cassels [3] and Kneser's papers [7, 8]. Let \( V \) be an \( n \)-dimensional vector space over a global field \( k \) or over its localization \( k_p \); let \( \varphi \) be a symmetric regular bilinear form on \( V \), \( f(x) = \varphi(x, x) \) the corresponding quadratic form, \( O(V) \) the group of linear
transformations of $V$ that keep the form $f$, $O^+(V)$ the group of transformations from $O(V)$ with determinant +1, $\theta$ the spinor norm homomorphism, $O'(V) = O^+(V) \cap \text{Ker}(\theta)$ the subgroup of autometries from $O^+(V)$ with spinor norm 1, $\nu_p(a)$ the order of $p$ at $a$ (see [9]). In [2], we introduce the notion of generalized quadratic lattices in $V$ and the notion of relative lattices, closely connected with them. Among generalized lattices the so-called generalized translations are of main interest; they correspond to the simplest relative lattices of the kind $M/L$, $M \supseteq L$, where $M$ and $L$ are the usual (classical) lattices in $V$, i.e., full finitely generated $\mathfrak{o}$- or $\mathfrak{o}_p$-modules (where $\mathfrak{o}$ is the ring of integers for the field $k$). The lattice $M/L$ may be regarded as an ordered set of residue classes from the quotient group $M/L$; the group $O(V)$ acts on the set of all generalized lattices; in particular, the group of autometries of a relative lattice $M/L$ can be defined as

$$O(M/L) = \{ \sigma \in O(V) \mid \sigma x \in x + L \text{ for all } x \in M \}.$$ 

It is shown in [2] that, if a relative lattice $M/L$ corresponds to a generalized lattice $L$, then the group of autometries $O(L)$ of this generalized lattice coincides with $O(M/L)$, and the classes, spinor genera, and $cg$-genera in the genus of $L$ are in one-to-one correspondence with the classes, spinor genera, and $cg$-genera in the genus of $M/L$. Therefore, to avoid the introduction of new (and rather complicated) definitions, we shall further formulate all the results in terms of relative lattices, keeping in mind that all of them may be literally carried over to generalized translations.

For a classical lattice $L \subseteq V$ and a vector $x \in V$, we denote $\varphi(x, L) = \{ \varphi(x, y) \mid y \in L \}$, $f(L) = \{ f(y) \mid y \in L \}$. In the usual fashion, we define for $L$ its norm $nL$ as the ideal generated by $f(L)$, its scale $sL = \{ \varphi(x, y) \mid x, y \in L \}$, and its volume $vL$ (see [9]); we also define the reduced volume $\mathfrak{d}L = sL^{-2}nL$ and the reduced scale $\mathfrak{g}L = (sL \cdot sL^\#)^{-1}$, where $L^\#$ is the dual of $L$. If $M/L$ is a relative lattice, then its norm, scale, volume, reduced volume, and reduced scale are defined as the norm, scale, volume, reduced volume, and reduced scale of $L$, respectively. We also define the order of inhomogeneity $i(M/L)$ of a relative lattice as the rank of $M/L$ regarded as a module, and the module of inhomogeneity $m(M/L) = \{ a \in k \mid aM \subseteq L \}$.

Throughout the paper, except for Theorem 3, we shall consider only local lattices, i.e., $\mathfrak{o}_p$-lattices from a quadratic space $V$ over a local field $k_p$ with $p \not| 2$. In particular, for such lattices we have $nL = sL$.

For any vector $x \in V$ with $f(x) \not= 0$ we may define a symmetry $\tau_x \in O(V)$ of the quadratic space $V$:

$$\tau_x(y) = y - 2 \frac{\varphi(x, y)}{f(x)} x,$$

and for it $\det(\tau_x) = -1$ and $\theta(\tau_x) = f(x)k_p^{2}$ (see [9, 3, 8]).

We denote by $T(V)$ and $T(M/L)$ the sets of all symmetries of the quadratic space $V$ and the quadratic lattice $M/L$:

$$T(V) = \{ \tau_x \mid x \in V, f(x) \not= 0 \} \subseteq O(V),$$

$$T(M/L) = T(V) \cap O(M/L),$$

and by $O'(M/L)$ the set of proper autometries of $M/L$ with spinor norm equal to 1:

$$O'(M/L) = O'(V) \cap O(M/L).$$

We also denote

$$T(M/L) = \theta(T(M/L)),$$

and so $T(M/L)$ is the subset of the group $k_p^{*}/k_p^{*}$ consisting of the spinor norms of symmetries from $O(M/L)$. The main purpose of this paper is to prove Theorem 1 below and Theorem 2, which is equivalent to it.

**Theorem 1.** Let $M/L$ be a relative $\mathfrak{o}_p$-lattice, $p \not| 2$. Then the group $O(M/L)$ is generated by the set $T(M/L) \cup O'(M/L)$, i.e., by the symmetries contained in it and by autometries of spinor norm 1.

**Remark.** It may be seen from the proof of Theorem 1 that in the case $m(M/L) \supseteq p$, i.e., for $M \subseteq p^{-1}L$, it is sufficient to take in Theorem 1 the set consisting of products of two or four symmetries of special kind in place of the group $O'(M/L)$. Slightly refining the proof of Theorem 1, it is not difficult to remove the above restriction on $m(M/L)$. We shall not consider these questions because Theorem 1 is quite sufficient for the theory of spinor genera.