THE ALGEBRA OF GENERALIZED JACOBI FIELDS

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We study the structure of those vector fields on the tangent bundle of an arbitrary smooth manifold which commute with the geodesic vector field defined by an affine connection. The study is restricted to polylinear fields generated by a pair of symmetric pseudotensor fields of type \((k, 1)\) and \((k+1, 1)\), \(k \geq 0\), defined on the manifold. We establish an isomorphism between the space of infinitesimal automorphisms of fixed type and the space \(h_k\) of the solutions of a partial differential equation generalizing the Jacobi equation for the infinitesimal automorphisms of the connection. It is shown that the spaces \(h_k\) are finite-dimensional and form a graduated Lie algebra \(\mathfrak{h} = \bigoplus_{k=0}^{\infty} h_k\). These algebras are classified in the case of one-dimensional manifolds. It is proved that if the geodesic vector field is complete, then so are the automorphisms corresponding to covariant constant fields of type \((1, 1)\).

INTRODUCTION

Let \(M\) be a connected \(n\)-dimensional \(C^\infty\)-manifold, and let \(TM\) be its tangent bundle. A vector field \(Y\) on \(TM\) in the standard map \((x, v)\) has the form

\[
Y(x, v) = Y_i^i(x, v) \frac{\partial}{\partial x^i} + Y_j^j(x, v) \frac{\partial}{\partial v^j}.
\]

(Here and below we use the agreement of summation over twice repeated indices (“Einstein’s rule”).) We call \(Y\) a polylinear field of rank \(k\), \(k \geq 0\), if its first and second components are homogeneous in \(v\) with degree \(k\) and \(k+1\), respectively, i.e., the field has the form

\[
Y(x, v) = a_{i_1...i_k}^i(x) v^{i_1} \cdots v^{i_k} \frac{\partial}{\partial x^i} + q_{i_1...i_{k+1}}^i(x) v^{i_1} \cdots v^{i_{k+1}} \frac{\partial}{\partial v^i},
\]

(1)

where \(A = \|a_{i_1...i_k}^i(x)\|\) and \(Q = \|q_{i_1...i_{k+1}}^i(x)\|\) are pseudotensor fields on the manifold \(M\). Having in mind Eq. (1), we write \(Y = Y_{A,Q}\).

An example of a polylinear field of rank one is given by the geodesic vector field \(S = SE_\omega\) corresponding to an affine connection on \(M\) with form \(\omega\), where \(E = \|\delta_j^i\|\) is the identity tensor field of type \((1, 1)\) on \(M\) (see §2).

One of the aspects of studying the symmetry group of a geodesic vector field \(S\) consists in the investigation of the algebra of infinitesimal automorphisms of \(S\). In the present paper, we consider a certain subalgebra of that algebra, which consists of polylinear infinitesimal automorphisms of the field \(S\) whose components are subject to some relations, and study the structure of this subalgebra. This is the algebra of generalized Jacobi fields.

In §1, we study the space \(\mathfrak{g}\) of finite linear combinations of symmetric pseudotensor fields. We consider the symmetrized operations of contraction, differential, and differentiation along a field, that are defined on this space, as well as basic relations between these operations. In §2, the structure of polylinear infinitesimal automorphisms of the geodesic vector field of rank \(k\) is described (Theorem 2.2). In §3, we prove that for any symmetric connection \(\nabla\) on \(M\), the space \(h_k\) of Jacobi fields of rank \(k\) form a graduated Lie algebra

\[
\mathfrak{h} = \bigoplus_{k=0}^{\infty} h_k,
\]

and the dimensions of its components have polynomial growth (Theorem 3.3). It is also shown that the maximal dimension of the spaces \(h_k\) for given \(n\) and \(k\) is reached in the case of the space \(\mathbb{R}^n\) with canonical
connection (Theorem 3.2). In a number of cases, sharp estimates of the dimensions are obtained: for example, for two-dimensional manifolds we have
\[ \dim \mathfrak{h}_k \leq 2(k + 1)(k + 3) \]
(Theorem 3.4), and for \( n \)-dimensional manifolds we have
\[ \dim \mathfrak{h}_1 \leq n \sum_{s=1}^{n} s 2^{n-s+1} \]
(Theorem 3.5).

In §4, a complete classification of possible algebras \( \mathfrak{h} \) in the case of one-dimensional manifolds is given (Theorem 4.2). We use the fact that if the manifold with connection is homogeneous, i.e., the pseudogroup of local affine automorphisms of the connection is transitive, then the algebra \( \mathfrak{h} \) is isomorphic to the maximal trivial subrepresentation of the monodromy representation in the algebra of germs of solutions of Eq. (18) (see §2) at an arbitrary point of the manifold (Theorem 4.1).

The next step in the study of the automorphisms is studying the question of completeness of the fields representing the infinitesimal automorphisms. The question is partly solved by a theorem of Kobayashi [1], which implies that if the geodesic vector field is complete, then the subalgebra of \( \mathfrak{h} \) generated by the fields of type \((1,0)\) consists of complete fields. In §5, we prove under the same condition that the elements of the subalgebra generated by the covariant constant fields of type \((1,1)\) are complete, and obtain a criterion of completeness for elements of the subalgebra generated by the nondegenerate fields of type \((1,1)\) (Theorem 5.1).

§1. THE SPACE OF SYMMETRIC PSEUDOTENSOR FIELDS

A pseudotensor field of type \((1,k)\) on the manifold \( M \) is an object represented in each chart \( M \) by a collection of \( C^\infty \)-functions
\[ F = \left\| F_{i_1\ldots i_k}(x) \right\|, \quad i, i_s = 1, \ldots, n; \quad 1 \leq s \leq k, \quad k \geq 0, \]
that change by the tensor law under linear changes of coordinates.

The set \( \mathcal{F}_k \) of the pseudotensor fields of type \((1,k)\) is a vector space with respect to the component-wise addition and multiplication by scalars. Define
\[ \mathcal{F} = \bigoplus_{i=0}^{\infty} \mathcal{F}_i. \]

Examples of pseudotensor fields are given by tensor fields of type \((1,k)\) and the form \( \omega \) of an arbitrary affine connection; the transition formulas for \( \omega \) have the form
\[ \Gamma'_{i'i''} \frac{\partial x'^i}{\partial x''_i} \frac{\partial x'^i}{\partial x''_j} = \frac{\partial x'^i}{\partial x''_{i'}} \Gamma_{i'i''}^{i''} - \frac{\partial^2 x'^i}{\partial x''_i \partial x''_j}, \]
and so indeed \( \omega \) is a pseudotensor field of type \((1,2)\).

The operation of symmetrization by covariant indices
\[ \{ \cdot \} : \mathcal{F} \to \mathcal{F} \]
is defined on \( \mathcal{F} \). The components of the symmetrized pseudotensor field are given by
\[ \{ F \}_{i_1 \ldots i_k}^{'} = \frac{1}{k!} \sum_{\sigma(1,\ldots,k)} F_{i_{\sigma(1)} \ldots i_{\sigma(k)}}^i, \]
where the sum is taken over all permutations \( \sigma \) of the set \( \{ 1, \ldots, k \} \).

The space of symmetric fields (of fields of type \((1,k)\)) is denoted by \( \mathcal{S} \) (respectively, by \( \mathcal{S}_k \)). A series of operations on this space arises in the usual way.