THE SPACE-TIME FINITE ELEMENT METHOD
FOR PARABOLIC PROBLEMS *

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Abstract: Adaptive space-time finite element method, continuous in space but discontinuous in time for semi-linear parabolic problems is discussed. The approach is based on a combination of finite element and finite difference techniques. The existence and uniqueness of the weak solution are proved without any assumptions on choice of the space-time meshes. Basic error estimates in $L^\infty (L^2)$ norm, that is maximum-norm in time, $L^2$-norm in space are obtained. The numerical results are given in the last part and the analysis between theoretic and experimental results are obtained.

Key words: semi-linear parabolic equations; space-time finite element method; existence and uniqueness; error estimate

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Introduction

The equations we considered are as follows

\[ \begin{cases} u_t - \Delta u = f(u), & \Omega \times [0, T], \\ u \mid_{\partial \Omega} = 0, & \partial \Omega \times [0, T], \\ u(\cdot, 0) = u^0, & \Omega, \end{cases} \]

(1)

where $\Omega \subset \mathbb{R}^2$, the function $f(u)$ satisfies

\[ |f(u)| \leq c |u|, \quad \forall u \in C(\Omega). \]

(2)

And $f(u)$ is Lipschitz continuous, i.e. it satisfies

\[ |f(u) - f(v)| \leq L |u - v|, \quad \forall u, v \in C(\Omega), \]

(3)

where $L$ is Lipschitz constant, $c$ is a positive constant.

As is well-known, the solutions of the parabolic equations above may be complex in structures, and may be singular in finite time. Therefore, in order to obtain accurate approximate

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solutions, it is necessary to choose an efficient numerical method suitable for discontinuous cases. As far as space-time finite element method, continuous in space but discontinuous in time is concerned, finite element discretizations in both space and time were exploited. It not only unifies the variables in spacial and temporal fields but also can deals with discontinuous problems flexibly. More important, it admits different meshes in different, space-time slab which make it a high adaptive method. Adaptive space-time finite element method has two basic characteristics. The first is reliability that the error control should be guaranteed by the automatic choice of unstructured space-time meshes, and the error accumulation is reduced as small as possible in the computational process. The other is efficiency which indicates that the mesh constructed by the adaptive algorithm should be reasonably close to an optimal mesh, i.e. a mesh with as few degrees of freedom as possible.

The paper [1] discussed the space-time adaptive method for linear parabolic equations and error estimate in $L_2$ norm in space. In [2], the first analysis of the reliability and efficiency of the adaptive method were studied, and optimal error estimates in $L_c (L_2)$ and $L_c (L_{c2})$ norm were given. The method was expanded to generic nonlinear problems in [3]. The common features of these papers are as follows: The prior and posterior error estimates were based on the strong stabilities and error estimates of dual continuous or discrete problems. In the meantime, local error estimates and Galerkin orthogonalities were used. In addition, the assumption on the choice of space-time mesh was $h \geq c h^2$. This condition must be satisfied, though a kind of new mesh-dependent norm was defined in [4]. The approach, discontinuous space-time finite element method for Eq. (1) we considered here is similar to the one for Schrödinger equations in [5]. Taking full advantages of the combination of finite element and finite difference techniques and of useful properties of Lagrange interpolating polynomials at the Radau points of each time interval $I_n$, we prove the existence and uniqueness of the weak solution, give the error estimate in $L^\infty (L_2)$ norm, that is maximum-norm in time, $L^2$ norm in space. And our results are valid without any restrictive assumptions on the choice of space-time meshes.

In Section 1, some definitions and notations are introduced. In Section 2, the existence and uniqueness of weak solution are proved. And the error estimate in $L^\infty (L^2)$ norm of the finite element solution is given. Finally the numerical analysis for some simple examples is presented.

1 Definitions and Notations

In order to introduce the space-time finite element method for Eq. (1), we discretize the time interval $[0, T]$ firstly. Let $0 = t^0 < t^1 < \cdots < t^N = T$, $I_n = (t^n, t^{n+1}]$, time step $k_n = t^{n+1} - t^n$, $n = 0, 1, 2, \cdots, N - 1$. Define the space-time domain $Q := \Omega \times [0, T]$ and the space-time slab $S^n := \Omega \times I_n$. We associate a partition $\{\Gamma_{zn}\}$ of $S^n$, in which $\tau$ denotes the element of the partition, $e$ is the boundary of the element $\tau$. And $h_\tau$ stands for the diameter of the element $\tau$ and $h_n = \max_{\tau \in I_n} h_\tau, h = \max_{\tau} h_\tau$.

**Definition 1** For each time interval $I_n$, define the finite element space

$$S^n_h = \{ \chi \in H^0_0 (\Omega) : \chi |_\tau \in P_{r-1} (\tau), \tau \in \Gamma_{zn} \},$$

where $P_{r-1} (\tau)$ denotes the polynomials of degree $r - 1$ in the element $\tau$, $n = 0, 1, \cdots, N - 1$. 