Lie Symmetries of Quadratic Two-Dimensional Difference Equations

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We find all systems of first-order quadratic autonomous two-dimensional difference equations which have two linear Lie symmetries. Knowledge of these symmetries permits the systems to be integrated by a reduction procedure.

The identification of integrable systems for continuous or discrete equations is an important problem in applied mathematics. Discrete dynamical systems have been studied in many contexts in the recent years. They appear in discretization procedures of continuous systems, or, more naturally, in models described within a discrete space, for instance, in many biological systems. Two-dimensional continuous systems of first-order autonomous ordinary differential equations have no chaotic behavior; however, there are chaotic two-dimensional autonomous difference equations, Hénon's map, for example (Hénon, 1976). In many cases, discretization of completely integrable continuous systems also can exhibit chaotic behavior (Date et al., 1982).

Although integrable discrete systems have been known for decades (MacMillan, 1971), few systematic studies were undertaken in this direction (Hirota, 1979; Maeda, 1987; Grammaticos et al., 1991; Quispel and Sahadevan, 1993). We study here two-dimensional systems of first-order quadratic autonomous difference equations. These equations are discrete counterparts of Lotka–Volterra continuous systems, which are important in population dynamics (Gardini et al., 1987). We analyze the invariance of these discrete equations under a continuous group of symmetries for determining integrable cases. Maeda (1987) extended Lie's algorithm for finding symmetries of difference equations and constructed a procedure for making a reduction of

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an autonomous system if it has a Lie symmetry. Quispel and Sahadevan (1993) extended this method to nonautonomous systems.

We summarize the method formulated by Quispel and Sahadevan (1993). Consider a system of coupled difference equations

\[ x_{n+1}^i = \Phi^i(x_n^i, n) \]  

where \( \Phi^i \) are given functions and \( i = 1, \ldots, N \). The infinitesimal transformation

\[ x_n^{i'} = x_n^i + \epsilon \xi^i(x^i, n) \]  

is a symmetry transformation if

\[ \xi^i(\Phi(x_n^i, n), n + 1) = \sum_{k=1}^N \xi^k(x_n^i, n) \frac{\partial}{\partial x_k} (\Phi^k(x_n^k, n)) \]  

The symmetry operator is

\[ U = \sum_{i=1}^N \xi^i(x_n^i, n) \frac{\partial}{\partial x^i} \]  

We now apply a coordinate transformation \( y = y(x) \) in the system (1), which brings the system to the form

\[ y_{n+1}^i = \Psi^i(y_n^i, n) \]  

and the symmetry operator is changed to

\[ U = \sum_{i=1}^N \xi'^i \frac{\partial}{\partial y^i} \]  

where

\[ \xi'^i(y, n) = \xi^i(x, n) \frac{\partial}{\partial x^j} (y^j) \]  

If a Lie symmetry exists, we can choose a coordinate system where

\[ \xi'^i(y, n) = \delta^i_1 \]  

and where the symmetry operator takes the form

\[ U = \frac{\partial}{\partial y^i} \]  

In this case the conditions for symmetry (3) are written as

\[ \frac{\partial}{\partial y^i} (\Psi^i) = \delta^i_1 \]