ISHIKAWA ITERATIVE PROCESS IN UNIFORMLY SMOOTH BANACH SPACES*

HUANG Zhen-yu (黄震宇)

(Department of Mathematics, Nanjing University, Nanjing 210093, P R China)

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Abstract: Let $E$ be a uniformly smooth Banach space, $K$ be a nonempty closed convex subset of $E$, and suppose $T: K \to K$ is a continuous $\Phi$-strongly pseudocontractive operator with a bounded range. Using a new analytical method, under general cases, the Ishikawa iterative process $\{x_n\}$ converges strongly to the unique fixed point $x^*$ of the operator $T$ were proved. The paper generalizes and extends a lot of recent corresponding results.

Key words: Ishikawa iterative process; $\Phi$-strongly pseudocontractive operators; uniformly smooth Banach spaces

Let $E$ be a uniformly smooth Banach space, $K$ be a nonempty closed convex subset of $E$, and suppose $T: K \to K$ is a continuous $\Phi$-strongly pseudocontractive operator. Denote the dual space of $E$ by $E^*$. We denote by $J$ the duality map from $E$ to $2^E$ defined by

$$J(x) = \{f \in E^*: \langle x, f \rangle = \|x\|^2 = \|f\|^2\}. \tag{1}$$

It is well-known that if $E$ is uniformly smooth, then $J$ is single-valued and is uniformly continuous on any bounded subsets of $E$.

$T: K \to K$ is defined $\Phi$-strongly pseudocontractive if for all $x, y \in K$, there exists $j(x - y) \in J(x - y)$, and there exists a strictly increasing function $\phi: [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$, such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|) \|x - y\|, \forall x, y \in K. \tag{2}$$

Particularly, if $\phi(s) = ks$, $k \in (0, 1)$, then the $\phi$-pseudocontractive operator $T$ is the strong pseudocontractive operator of [1] and [2].

Lemma 1[1] Let $E$ be a uniformly smooth Banach space, $J(x)$ be a duality mapping, then for all $x, y \in E$, we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \forall j(x + y) \in J(x + y). \tag{3}$$

Recently, ZHOU Hai-yun (see Ref. [2], Theorem) proved that in the uniformly smooth Banach spaces, when $T$ is a continuous and strongly pseudocontractive operator, Ishikawa
iterative process strongly converges to the unique fixed point of \( T \). Meanwhile, DING Xie-ping (see Ref. [3], Theorem 3.2) proved that, in arbitrary Banach spaces, while \( T \) is a Lipschitz continuous and \( \Phi \)-strongly pseudocontractive operator, the Ishikawa iterative process strongly converges to the unique fixed point of \( T \). In 1999, ZHOU Hai-yun (see Ref. [4], Theorem 2.2) showed that, in uniformly smooth Banach spaces, if \( T \) is uniformly continuous \( \Phi \)-strongly pseudocontractive and the Ishikawa iterative process \( \{ x_n \} \) is bounded, then \( \{ x_n \} \) strongly converges to the unique fixed point of \( T \). In 1998, the author (see Ref. [5], Theorem 1) presented the strong convergence under more restrictions on the parameters \( \{ \alpha_n \}, \{ \beta_n \} \), if \( T \) is continuously \( \Phi \)-strong pseudocontraction. The purpose of this paper is to prove the following theorem by using a new analytical technique:

**Theorem** Let \( E \) be a uniformly smooth Banach space, \( K \subseteq E \) be a nonempty closed convex subset, \( T : K \to K \) be a continuous \( \Phi \)-strongly pseudocontractive operator with a bounded range. Suppose \( T \) has at least a fixed point \( x^* \) in \( K \). Suppose the parameters \( \{ \alpha_n \}, \{ \beta_n \} \) satisfying the following conditions:

\[
0 \leq \alpha_n, \beta_n < 1, \forall n \geq 1; \quad \sum_{n=1}^{\infty} \alpha_n = +\infty; \quad \lim_{n \to \infty} \alpha_n = 0, \quad \lim_{n \to \infty} \beta_n = 0. \tag{4}
\]

For arbitrary \( x_1 \in K \), the Ishikawa iterative process \( \{ x_n \} \) is defined by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \quad n \geq 1; \quad y_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 1. \tag{5}
\]

Then \( \{ x_n \} \) converges strongly to the unique fixed point \( x^* \) of the operator \( T \).

**Proof** From Ref. [5], we know that the fixed point \( x^* \in K \) is the unique fixed point of \( T \) in \( K \). By mathematical induction, we can prove that the sequences \( \{ x_n - x^* \}, \{ y_n - x^* \} \) and \( \{ Ty_n - Tx^* \} \) are all bounded subsets in \( K \). Set

\[
M = \sup \{ \| Tx - Ty \| : x, y \in K \} + \sup \{ \| x_n - x^* \| : n \geq 1 \} + \sup \{ \| y_n - x^* \| : n \geq 1 \}.
\]

Obviously \( M < +\infty \). From Lemma 1, we obtain that

\[
\| x_{n+1} - x^* \|^2 \leq \| x_n - x^* \|^2 + M^2\alpha_n^2 + 2\varepsilon_n\alpha_n + 4M^2\beta_n\alpha_n - 2\alpha_n\phi(\| y_n - x^* \| ) \| y_n - x^* \| , \tag{6}
\]

where

\[
e_n = \langle Ty_n - Tx^*, J(x_{n+1} - x^*) - J(y_n - x^*) \rangle.
\]

Notice that as \( n \to +\infty \),

\[
(x_{n+1} - x^*) - (y_n - x^*) = x_{n+1} - y_n = \beta_nx_n - \alpha_nx_n + \alpha_nTy_n - \beta_nTx_n \to 0,
\]

and \( E \) is a uniformly smooth Banach space, then \( J \) is uniformly continuous on any bounded subsets of \( E \), so we have

\[
\| J(x_{n+1} - x^*) - J(y_n - x^*) \| \to 0,
\]

hence \( \lim \varepsilon_n = 0 \). Denote \( \lambda_n = M^2\alpha_n + 2\varepsilon_n + 4M^2\beta_n \). Clearly \( \lim \lambda_n = 0 \). So from (6), we know

\[
\| x_{n+1} - x^* \|^2 \leq \| x_n - x^* \|^2 + \lambda_n\alpha_n - 2\alpha_n\phi(\| y_n - x^* \| ) \| y_n - x^* \| \leq \langle 7 \rangle \| x_n - x^* \|^2 + \alpha_n\lambda_n - \phi(\| y_n - x^* \| ) \| y_n - x^* \| - \]