POISSON LIMIT THEOREM FOR COUNTABLE MARKOV CHAINS IN MARKOVIAN ENVIRONMENTS *

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Abstract: A countable Markov chain in a Markovian environment is considered. A Poisson limit theorem for the chain recurring to small cylindrical sets is mainly achieved. In order to prove this theorem, the entropy function h is introduced and the Shannon-McMillan-Breiman theorem for the Markov chain in a Markovian environment is shown. It’s well-known that a Markov process in a Markovian environment is generally not a standard Markov chain, so an example of Poisson approximation for a process which is not a Markov process is given. On the other hand, when the environmental process degenerates to a constant sequence, a Poisson limit theorem for countable Markov chains, which is the generalization of Pitskel’s result for finite Markov chains is obtained.

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1 The Process and Statement of the Result

As a dynamical counterpart of the classical Poisson limit theorem for 0-1 valued independent random variables, a Poisson limit theorem is proved by Pitskel[1,2] for ergodic finite Markov chains, for strongly ergodic non-homogeneous finite Markov chains. In this paper, we consider a countable Markov chain in a Markovian environment. We mainly prove a Poisson limit theorem for the chain recurring to small cylindrical sets. It’s well-known that a Markov process in a Markovian environment is generally not a standard Markov chain, so we here give an example of Poisson approximation for a process which is not a a Markov process. On the other hand, when the environmental process degenerates to a constant sequence (i.e., the state space of the environmental process consists of only one point), we obtain a Poisson limit theorem for countable Markov chains, which is the generalization of Pitskel’s result for finite Markov chains[3].

Let \( \mathbb{Z}_+ \) be the set of all positive integers, and let \( \mathbb{Z} = \{0\} \cup \mathbb{Z}_+ \). Let \( \Theta \) be a countable...
space, \( \{ P(\theta) : \theta \in \Theta \} \) be transition probabilities on a countable state space \( E \). Let \( \tilde{X} = \{ X_0, X_1, \cdots \} \) be a stochastic sequence taking values in \( E \), \( \tilde{\xi} = \{ \xi_0, \xi_1, \cdots \} \) be a Markovian stochastic sequence taking values in \( \Theta \) and having transition probabilities \( K(\theta, \alpha), \theta, \alpha \in \Theta \). For \( n \in \mathbb{Z} \) and \( x \in E \), if \( \tilde{X} \) and \( \tilde{\xi} \) are satisfied by
\[
P(X_{n+1} = x \mid X_0, X_1, \cdots, X_n; \xi_0, \xi_1, \cdots) = P(\xi_n; X_n, x), \quad a.s.,
\]
then \( \tilde{X} \) is called a Markov chain in a Markovian environment and \( \tilde{\xi} \) is called a Markovian environmental process\(^4\).\(^5\).\(^6\).

It is well-known that the bivariate process \((X, \xi) = \{(X_0, \xi_0), (X_1, \xi_1), \cdots \} \) is a standard Markov chain\(^6\). We call it a bi-chain. The transition probabilities of the bi-chain are determined by
\[
P((x, \xi), (y, \alpha)) = K(\theta, \alpha)P(\theta; x, y), \quad \theta, \alpha \in \Theta, x, y \in E.
\]
We let \( E^Z, \Theta^Z \) and \( \Omega = (E \times \Theta)^Z \) denote respectively the space of trajectories of the process \( \tilde{X} \), the space of trajectories of the process \( \xi \) and the space of trajectories of the bi-chain.

Throughout the paper, we always assume that the bi-chain is irreducible and aperiodic. Let \( \bar{P} \) be the Markov measure of the bi-chain with stationary distribution \( \pi = \{ \pi(x, \theta) : (x, \theta) \in E \times \Theta \} \), \( P^* \) be the marginal distribution of \( \bar{P} \) with respect to the marginal process \( \tilde{X} \). Let \( T \) denote the shift transformation on \( E^Z \).

For each \( n \in \mathbb{Z} \) and \( x \in E^Z \), let
\[
S_n(x) = \{ y \in E^Z : y_i = x_i, \, 0 \leq i \leq n \}.
\]
For \( \mu > 0 \), and \( \{ a(n) \} \) being a real number sequence such that \( \lim a(n) = 0 \), let
\[
N_n = [(\mu + a(n))P^*(S_n(x))].
\]
Our main result is stated as follows.

**Theorem** Assume that the bi-chain is \(*\)-mixing\(^7\) and irreducible and aperiodic. Either
\( (i) \) \( \Theta \) is finite space, or
\( (ii) \) \( \inf \{ P(\theta, x, y) > 0 : \theta \in \Theta, x, y \in E \} > \rho \) for some constant \( \rho > 0 \),
then for almost all \( \tilde{x} \in E^Z \) (with respect to \( P^* \))
\[
\sum_{i=0}^{n} I_{S_i(\tilde{x})} ^n \circ T^n(\tilde{X})
\]
converges weakly to the Poisson distribution with respect to parameter \( \mu \), that is
\[
\lim_{n \to \infty} P^* \left( \sum_{i=0}^{n} I_{S_i(\tilde{x})} ^n \circ T^n(\tilde{X}) = k \right) = \frac{\mu^k}{k!} e^{-\mu} \quad (k = 0, 1, \cdots).
\]

Noting that the process \( \tilde{X} \) becomes a standard Markov chain when the state space \( \Theta \) of the environmental process consists of only one point, we obtain a Poisson limit theorem for countable Markov chains as the corollary of this theorem.

The theorem will be proved by checking the conditions of Sevast'yanov's theorem\(^3\) stated in Section 3, and that the proof of the theorem can be completed mainly relies on the lemmas established in Section 2. It is mentioned that the lemmas are themselves interesting.