1 Introduction

Wave propagation in compliant ducts filled with streaming fluids is of basic interest in fluid mechanics (LiLTtiLL, 1978; ShAPIRo, 1977; KAMM and ShAPIRo, 1979; CANCELLI and PEDLEY, 1985) and also has several biological implications, particularly in vascular pulse propagation (STiETT3R et al., 1963; JONES, 1969; AnliKER et al., 1971; PEDLEY, 1980; ROOZ et al., 1985) and expiratory flow limitation (DAWSON and ELLIOTT, 1977; HYATT et al., 1979; MINK, 1984; PEDERSEN and INGRAM, 1985; O'DONNELL et al., 1986).

The problem is to determine in a coupled fluid-compliant tube system the relationship between pressure and flow velocity perturbations, as well as the speed at which these perturbations propagate (wave speed). When the flow velocity is not much smaller than the wave speed the convective acceleration terms of the fluid momentum equations have to be considered and the problem becomes nonlinear.

Not purely numerical solutions usually rely upon the one-dimensional method of the characteristics, so that the unknown axial flow velocity profile is assumed to be rectangular. This one-dimensional flow simplification is used in all the above-mentioned publications, but it has apparently not been firmly validated. The purpose of this study is therefore to investigate the impact of the axial flow velocity profile on this wave propagation phenomenon.

2 Basic equations

The wave motion to be discussed is assumed to show axial symmetry in the \( x \) co-ordinate system, where the \( x \)-axis is identical with the tube axis and \( r \) is the radial co-ordinate normal to that direction. No field forces act on the inviscid and incompressible fluid. The wavelength of the perturbations is much larger than the tube radius, which allows the radial pressure dependence to be neglected. The equations governing the fluid motion are the \( x \)-momentum and the continuity equations

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial x} \tag{1}
\]

\[
\frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (rV) = 0 \tag{2}
\]

The unknowns \( U(x, r, t), V(x, r, t) \) and \( P(x, t) \) are the \( x \)-and \( r \)-velocity components and the pressure of the flow, \( t \) is the time and \( \rho \) is the fluid density.

The uniform, circular tube is assumed to be thin walled, surrounded by a constant pressure and not submitted to longitudinal stresses. The equations for the motion of the tube reduce then to the simple ‘tube law’

\[
A = A[P(x, t)] \tag{3}
\]

where \( A = \pi R^2 \) is the cross-sectional area of the tube and \( R \) its radius.

The boundary condition \( V \to 0 \) as \( r \to 0 \) expresses the axial symmetry, whereas \( V = \partial R/\partial t + U \partial R/\partial x \) for \( r = R \) implies the impermeability of the tube wall. Multiplying this equation by \( 2\pi R \) we obtain the boundary condition

\[
2\pi RV = \frac{\partial A}{\partial t} + U \frac{\partial A}{\partial x} \quad \text{for} \quad r = R \tag{4}
\]

Before solving the system of eqns. 1–4 by a linearised method the nonlinear theory will be briefly outlined.
3 Nonlinear theory

In the quasi-one-dimensional, nonlinear theory proposed by Fox and Sibiel (1965) and by Barnard et al. (1966a; b) the axial flow velocity profile is not a priori specified. This theory, which includes the classical one-dimensional theory as a particular case, will therefore be presented here. Multiplying eqns. 1 and 2 by $r$, integrating them from 0 to $R$ and considering eqn. 4 one obtains the integral form of the momentum and continuity equations

$$\frac{d\bar{U}}{dt} + \frac{U}{A} (1 - B) \frac{dA}{dt} + BU \frac{d\bar{U}}{dx} + \bar{U}^2 \frac{dB}{dx} = -\frac{1}{\rho} \frac{dP}{dx}$$

(5)

$$\frac{dA}{dt} + \frac{\partial}{\partial x}(A\bar{U}) = 0$$

(6)

The cross-sectional mean velocity $\bar{U}$ and the abbreviation $B$ are defined as

$$U(x, t) = \frac{2}{R^2} \int_0^R U_r \, dr$$

(7)

$$B(x, t) = 2/(R\bar{U}) \int_0^R U^2 r \, dr$$

(8)

The four unknowns $\bar{U}$, $A$, $P$ and $B$ are inter-related only by three equations, eqns. 3, 5, 6 and, furthermore, the derivative $\partial B/\partial x$ appears in eqn. 5. In consequence, the method of the characteristics cannot be directly applied to this equation system. Therefore, a supplementary equation is needed which is obtained by assuming $B$ constant (Fox and Sibiel, 1965; Barnard et al., 1966a; b; Skalak, 1972). This can be achieved by introducing the variable $\Phi$ and $\Psi$.

$$U(x, t) = \Phi(r) \, e^{ia(x - ct)}$$

$$P(x, t) = \bar{P} \, e^{ia(x - ct)}$$

(17)

where $\Phi$, $\Psi$, $\Phi(r)$ and $\bar{P}$ are functions of $x$ and $t$.

4 Linearised theory

The system considered now is that of a steady basic state on which small amplitude perturbations are superposed.

$$U(x, t) = U_o(r) + u(x, t)$$

(14)

$$V(x, t) = 0 + v(x, t)$$

(15)

$$P(x, t) = P_o + p(x, t)$$

(16)

$U_o(r)$ is the known axial velocity profile of the basic flow, $P_o$ the corresponding constant pressure and $u$, $v$, $p$ the perturbations.

Let $u = 1/r \partial \Psi/\partial r$ and $v = -1/r \partial \Psi/\partial x$ derive from the stream function $\Psi$, so that the continuity equation is automatically satisfied. We shall seek for travelling wave solutions of the form

$$\Psi(x, r, t) = \Phi(r) \, e^{ia(x - ct)}$$

(17)

where $a$ is the wave number and $c$ is the propagation speed of the perturbations. Introducing eqns. 14–16 into the fluid momentum equation (eqn. 1) and in the boundary condition of eqn. 4, replacing $u$ and $v$ by their $\Psi$-definition and considering eqns. 17 we obtain after linearisation

$$\frac{d\Phi}{dr} - \frac{dU_o}{dr} \Phi = -\bar{P}/\rho$$

(18)

$$-2\pi\Phi = \frac{dA}{dr} (U_s - c)\bar{P} \quad \text{for} \quad r = R_0$$

(19)

$R_0$ is the undisturbed tube radius and $U_s = U_o(R_0)$ is the velocity at which the basic flow slips along the undisturbed wall. The solution of eqn. 18 is

$$\Phi = -\bar{P}(U_o - c)/\rho \int_0^r s \, ds \quad \text{for} \quad r = R_0$$

(20)

This integral constant being zero because $\Phi/r \to 0$ as $r \to 0$. Eliminating $\Phi/\bar{P}$ between eqn. 20 with $r = R_0$ and eqn. 19 we obtain the eigenvalue relationship for the unknown wave speed $c$

$$\frac{1}{a^2} = \frac{2}{R_0^2} \int_0^{R_0} \frac{r \, dr}{(U_o - c)^2}$$

(21)

4.1 A general property of eqn. 21

We now investigate some properties of eqn. 21 for general profiles $U_o(r)$. First, the left hand side of this equation as well as $R_0$ and $r$ on the right-hand side are real and positive. Also, $U_o(r)$ being real, an elementary complex number analysis shows that $c$ is real: in the long wavelength approximation, the present inviscid perturbation problem shows indifferent stability. Secondly, eqn. 21 has no solution in the interval $(U_{min}, U_{max})$, where $U_{min}$ is the minimum and $U_{max}$ the maximum of $U_o(r)$ for $0 < r < R_0$. This follows from the nonexistence (divergence) of the integral in eqn. 21 when an essential singularity such as $x^{-2}$ lies in the integration range. Thirdly, the integral in eqn. 21 increases monotonically from 0 to $+\infty$ when $c$ varies from $-\infty$ to $U_{min}$ and decreases monotonically from $+\infty$ to 0 when $c$ increases from $U_{max}$ to $+\infty$. In consequence, eqn. 21 always has the two solutions $c_-$ and $c_+$, and only these two, which satisfy $-\infty < c_- < U_{min}$ and $U_{max} < c_+ < +\infty$.

4.2 Quadratic velocity profiles

It does not seem possible to give simple explicit solutions of eqn. 21 for arbitrary velocity profiles $U_o(r)$. However, the quadratic profile

$$U_o(r) = 2(\bar{U}_0 - U_o)(1 - r^2/R_0^2) + U_s$$

(22)