Coherent States and Infinite-Dimensional Lie Algebras: an Outlook(*)(**).

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Summary. — The possible extension of the notion of generalized coherent state to the case of infinite-dimensional affine Lie algebras is discussed with special attention to the resulting topological structure of the coherent states manifold, and to its connection with the structure of the algebra. The relevance for the solution of nonlinear dynamical systems equations of motion is briefly reviewed.

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The generalized coherent states associated with an arbitrary Lie group \( G \) constitute an overcomplete set of quantum states, labelled by a point in a Kählerian manifold \( \mathcal{M} \) homogeneous under the action of \( G \).

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Typically $\mathcal{M}$ is the homogeneous factor space quotient of $G$ by the stability subgroup $K$ leaving some vector $|\omega\rangle$ in the Hilbert space of states $\mathcal{H}$ fixed.

If $G$ is a connected, simply connected nilpotent Lie group, its representations can all be obtained by Kirillov’s coadjoint orbit method (2), and $\mathcal{M}$ can be identified with the symplectic manifold constituted by an orbit of the coadjoint representation.

In fact, if $g^*$ is the space dual to the Lie algebra $g$ of $G$, namely the space of linear functionals on $g$, then $G$—which acts in $g$ by the adjoint representation $\text{Ad}(g)$, $g \in G$—acts in $g^*$ by the coadjoint representation $\text{Ad}^*(g)$, and $g^*$ is foliated into orbits under such an action.

It was shown by Kirillov that any homogeneous symplectic manifold homogeneous with respect to the action of $G$ is locally isomorphic with an orbit of the coadjoint representation of $G$; moreover, it derives from the Borel-Weil-Bott theory that these orbits are, for compact semi-simple Lie groups, in one-to-one correspondence with nonequivalent unitary representations of the group itself.

Let us further recall that generalized coherent states are defined by

\begin{equation}
|\zeta\rangle = |\zeta_0\rangle = \pi^{-1}(g) T(g) |\omega\rangle,
\end{equation}

where $T(g)$, $g \in G$, are the holomorphic representations of the complexification of $G$, and $\pi(g)$ is a holomorphic character for all $g$'s in the coset labelled by the point $\zeta \in G/K \sim \mathcal{M}$; and that $\mathcal{M}$ is locally isomorphic to $\mathbb{C}^n$ for some integer $N \geq 1$.

On the other hand, if $G$ is the dynamical group of a physical system, whose space of states is $\mathcal{H}$, then the time evolution of a state initially represented by a point $\zeta_0 \in \mathcal{M}$ is a path entirely embedded in $\mathcal{M}$ (more precisely, $G$ needs not to be strictly a dynamical group; what is required is that the system Hamiltonian $H$ be coherence preserving (4) for the coherent states of $G$).

The quantum evolution path in $\mathcal{M}$ is defined by the Feynman propagator

\begin{equation}
\langle \zeta'' | \zeta' | \zeta'\rangle = \langle \zeta'' \rangle \exp \left[ -\frac{i}{\hbar} (t'' - t') H \right] |\zeta'\rangle,
\end{equation}

which can be written in the path integral form

\begin{equation}
\langle \zeta'' | \zeta' | \zeta'\rangle = \int D[\zeta(t)] \exp \left[ \frac{i}{\hbar} S[\zeta(t)] \right]
\end{equation}

with the action functional given by

\begin{equation}
S[\zeta(t)] = \int L dt = \int \langle \zeta(t) | (i\hbar \partial_t - H) | \zeta(t)\rangle dt.
\end{equation}
