STABILITY OF NAVIER-STOKES EQUATION(II)

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Abstract

In this paper, we make some comparisons between the solutions for Navier-Stokes equation and those for heat conduction equation.

In his study of Navier-Stokes equation, professor J. Leray, a French mathematician and authority on partial differential equation, starting from heat conduction equation, obtained some results of properly posed of certain initial boundary value problems of Navier-Stokes equation. Professor R. Temam of University de Paris XI and other experts in this field also brought up the possibility of comparing these two classes of equations. This paper attempts to describe and prove the fundamental difference between these two.

Key words solution space, Janet number, Navier-Stokes equation

I. Review: The Relationship of Two Kinds of Cauchy Problem of Heat Conduction Equation

In $\mathbb{R}^n \times \mathbb{R}^+$, the simplest parabolic equation, that is, the so-called heat conduction equation goes as follows:

$$\frac{\partial U}{\partial t} - a^2 \Delta U = f(x, t)$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$, $a \in \mathbb{R}$, constant.

Usually there are two kinds of initial values problem for this equation:

1. Cauchy problem on hyperplane $\{t = 0\}$

$$\frac{\partial U}{\partial t} - a^2 \Delta U = f(x, t), \quad U(x, t) |_{t=0} = \varphi(x)$$

The solution to this problem is

$$U(x, t) = \int_{\mathbb{R}^n} \varphi(\xi) U(x - \xi, t) d\xi + \int_0^t \int_{\mathbb{R}^n} f(\xi, \tau) U(x - \xi, t - \tau) d\xi d\tau \quad (1.1)$$

where

$$U(x, t) = \frac{1}{(2a)^n \pi^n t^{n/2}} \exp \left[ - \frac{|x|^2}{4at} \right]$$

It is already proved that if $\varphi$ is continuous and bounded, then the solution is unique in the class of bounded functions, and is continuously depend on initial condition. Besides, for any open set $O$ in $\mathbb{R}^n \times \mathbb{R}^+, U(x, t)$ belongs to $C^\infty(O)$ and is analytic in $O \cap \{t = t_0\}$.  

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If we weaken the condition of continuity and boundedness of $\varphi(x)$ to $\varphi(x) \in M_\sigma(T)$ ($0 < \sigma \leq 2$), the above conclusion still remains valid. Whereas when $\sigma > 2$, the uniqueness of its solution is no longer valid as was shown in an example given by Tikhonov. $M_\sigma(T)$ is therefore the biggest function set which ensures the uniqueness of the solution for problem (1).

Apart from this, hyperplane $\{t = 0\} \subset \mathbb{R}^{n+1}$ is the characteristic surface of heat conduction equation. It boils down to this: the classic theory proves the existence uniqueness and stability of the analytic solutions for heat conduction equation. While (1.1) correctly presents the expression of this solution.

To facilitate illustration of the problem (without losing generality), we take $n = 1$. Assume $a = 1$. $u(x, t) = 0$. Problem (1)' now becomes as follows:

\[
\begin{cases}
\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0 \\
U(x, t) |_{t=0} = \varphi(x) \\
(x, t) \in \mathbb{R} \times \mathbb{R}_+
\end{cases}
\]

while (1.1) becomes:

\[
U(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\xi) \exp \left[ -\frac{(x-\xi)^2}{4t} \right] d\xi
\]

where

\[
g(\xi, x, t) = \frac{1}{\sqrt {\pi t}} \exp \left[ -\frac{(x-\xi)^2}{4t} \right]
\]

2° Cauchy problem on $\{x = 0\}$.

\[
\begin{cases}
\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0, \\
U(x, t) |_{x=0} = \psi(t), \\
\frac{\partial U}{\partial x} |_{x=0} = \omega(t)
\end{cases}
\]

We assume that $\psi(t) \in C^\infty$. According to Cauchy-Kowalevskia theorem, the following series:

\[
u(x, t) = \psi(t) + x\omega(t) + \sum_{n=2}^{\infty} \frac{x^n}{n!} \theta_n(t)
\]

is the unique stable analytic solution for (2), where $\theta_n(t)$ is determined by

\[
\begin{align*}
\theta_2(t) & = \psi'(t), \\
\theta_3(t) & = \omega'(t), \\
\theta_{n+1}(t) & = \theta_n'(t) \quad (n \geq 2)
\end{align*}
\]

3° The relationship between two kinds of Cauchy problem.

As described in (1 °), $u(x, t)$, determined by (1.2), belongs to $C^\infty(\mathbb{R} \times \mathbb{R}_+)$. and for any $t > 0$, $u(x, t)$ is an analytic function on $\mathbb{R}$. So we can calculate as follows:

\[
\begin{align*}
U(x, t) |_{x=0} & = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \varphi(\xi) g(\xi, 0, t) d\xi \\
\frac{\partial U}{\partial x} |_{x=0} & = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \varphi(\xi) g'(\xi, 0, t) d\xi
\end{align*}
\]

\[M_\sigma(T) = \{U(x, t), 0 \leq t \leq T \mid \exists A_U > 0, k_U > 0, \}
\]

\[|U(x, t)| \leq A_U \cdot \exp(k_U|x|^{\sigma}) \quad (\sigma \geq 0)
\]