GENERALIZED BIHARMONIC OPERATOR AND ITS APPLICATION TO
THE BENDING OF ELASTIC THIN PLATES

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Abstract

In this paper, δ-function is used to construct the generalized biharmonic operators, the corresponding quadratic function is presented, and the latter is applied to the bending of elastic thin plates. The result shows that when the arguments in the variational functional are generalized functions, discontinuity to some degree is allowed, and the modified variational principle by using the Lagrange multipliers is merely a special form of the result mentioned above.

Key words  δ-function, generalized derivative, thin plate, variation
(mathematics)

Finite element method for linear differential equations is a numerical approach to sectional polynomial by converting operator equations to equivalent variational principles. For the discontinuity to some degree of the arguments in quadratic functional resulted from piecewise interpolation. Fraeijs de Veubeke (1974)[10], J. T. Oden and J. N. Reddy (1976)[23], and Hu Hai-chang (1981)[15] pointed out respectively that the arguments in variational principle should be considered as generalized functions. For evading the concept of generalized function, the traditional way is to convert the classical operator equations to its equivalent variational principles and to modify it with Lagrange multipliers[10,15] then. Another way is to adopt certain conversion to evade generalized functions, in which the modified generalized variational principle is considered as a special form of the generalized variational principle that is convenient for its application in finite element method without concerning generalized function, but not considered as further extension of the generalized variational principle[13]. Thus, the purpose of this paper is to proceed directly the generalized differential equations to found a relevant variational principle in which the arguments are generalized functions. The concept of generalized differential equations has been discussed in [2] and [6]. But in this paper, the method given in [7] is used to define the discontinuous field variables function sectionally, the concrete mathematical model of generalized biharmonic differential operators acting on generalized function is presented on the concept of generalized derivative. For convenience of the application of generalized function to engineering, no more abstract functional spaces are introduced. In addition, the form of quadratic functional is presented and applied to the bending of elastic thin plates.
1. Generalized Biharmonic Operators

Suppose that a two-dimensional closed domain \( \mathcal{D} = \Omega \cup \tilde{\Omega} \) and \( \partial \Omega = \Gamma_a \cup C_i \), where \( \Gamma_a \) is the outer boundary of \( \Omega \), \( C_i \) is the encirclement line caused by the discontinuity to some degree of the field variable function in \( \Omega \). Then assume that \( W = W(x, y) \) is a generalized function defined in \( \Omega \), which has the generalized differential instead of the differential in its ordinary sense, that is

\[
\langle D_{xxyy}W, \phi \rangle = \langle W, \phi_{xxyy} \rangle \quad \forall \phi(x, y) \in C^2_a(\Omega) \tag{1.1}
\]

where \( \phi_{xxyy} = \frac{\partial^4 \phi(x, y)}{\partial x^4} \). \( \partial_{xxyy} \) is the fourth order generalized partial derivative with respect to the variable \( x \). Noticed that \( W(x, y) \) is a differentiable sectional continuous function except on \( C_i \). By integration by parts and by summation, we have

\[
\langle W, \phi_{xxyy} \rangle = \langle W_{xxyy}, \phi \rangle + \langle [W_{x}], \phi \rangle = \langle [W_{x}], \phi \rangle + \langle [W], \phi_{x} \rangle \tag{1.2}
\]

where \( \langle \cdot, \cdot \rangle_{\partial \Omega} = 0 \) was already taken into account. \( \langle \cdot, \cdot \rangle_{\partial \Omega} \) represents the inner product along the boundary, and \( l = \cos(n, x) \) is the direction cosine of \( x \) along the outward normal line \( n \) of curve \( C_i \). Besides

\[
[W] = W(x^i, y^i) - W(x^i, y_i) = W(C^i) - W(C_i) \\
[W_x] = \frac{\partial W(C^i)}{\partial x} - \frac{\partial W(C_i)}{\partial x}, \quad [W_{xx}] = \frac{\partial^2 W(C^i)}{\partial x^2} - \frac{\partial^2 W(C_i)}{\partial x^2}
\]

(1.3)

Using the properties of Dirac \( \delta \)-function, equation (1.2) may be expressed as

\[
\langle D_{xxyy}W, \phi \rangle = \langle W_{xxyy}, \phi \rangle + \langle [W_{x}], \phi \rangle = \langle [W], \phi_{x} \rangle + \langle [W], \phi_x \rangle \tag{1.4}
\]

where point \((x^i, y^i)\) is on the discontinuous line \( C_i \). Hence

\[
D_{xxyy}W = W_{xxyy} + [W_{x}] \partial_x \delta(x-x^i) \delta(y-y^i) \\
+ [W_x] \partial_x l \frac{\partial \delta(x-x^i) \delta(y-y^i)}{\partial x} + [W_{xx}] \partial_x l \frac{\partial^2 \delta(x-x^i) \delta(y-y^i)}{\partial x^2} \\
+ [W] \partial_x l \frac{\partial^3 \delta(x-x^i) \delta(y-y^i)}{\partial x^3} \tag{1.5}
\]

where \([\cdot]_{C_i}\) represent the discontinuous values along \( C_i \) (see equation (1.3)).

Similarly, we can get the corresponding expressions of \( D_{yyxx}W \), \( D_{xyyx}W \) and \( D_{yxxy}W \). Consequently, we can write the generalized biharmonic operator as

\[
\Delta^2_{g}W = \Delta^2W + \{ [W_{xxyy}]_{C_i} l + [W_{xx}]_{C_i} m + [W_{yy}]_{C_i} l \} \\
+ [W_{xy}]_{C_i} m \delta(x-x^i) \delta(y-y^i) + \{ [W_{xx}]_{C_i} l \}
\]